In August 2003, the planet Mars passed closer to Earth than it had in almost 60,000 years. Like Earth, Mars rotates on its axis and thus has days and nights. The photos here were taken by the Hubble telescope and show two nearly opposite sides of Mars. (Source: www.hubblesite.org) In Exercise 92 of Section 6.2, we examine the length of a Martian day.

Phenomena such as rotation of a planet on its axis, high and low tides, and changing of the seasons of the year are modeled by periodic functions. In this chapter, we see how the trigonometric functions of the previous chapter, introduced there in the context of ratios of the sides of a right triangle, can also be viewed from the perspective of motion around a circle.
6.1 Radian Measure

Radian Measure

In most applications of trigonometry, angles are measured in degrees. In more advanced work in mathematics, *radian measure* of angles is preferred. Radian measure allows us to treat the trigonometric functions as functions with domains of real numbers, rather than angles.

Figure 1 shows an angle θ in standard position along with a circle of radius r. The vertex of θ is at the center of the circle. Because angle θ intercepts an arc on the circle equal in length to the radius of the circle, we say that angle θ has a measure of 1 radian.

Radian

An angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of 1 radian.

It follows that an angle of measure 2 radians intercepts an arc equal in length to twice the radius of the circle, an angle of measure \( \frac{1}{2} \) radian intercepts an arc equal in length to half the radius of the circle, and so on. In general, if θ is a central angle of a circle of radius r and θ intercepts an arc of length s, then the radian measure of θ is \( \frac{s}{r} \).

Converting Between Degrees and Radians

The circumference of a circle—the distance around the circle—is given by \( C = 2\pi r \), where r is the radius of the circle. The formula \( C = 2\pi r \) shows that the radius can be laid off \( 2\pi \) times around a circle. Therefore, an angle of 360°, which corresponds to a complete circle, intercepts an arc equal in length to \( 2\pi \) times the radius of the circle. Thus, an angle of 360° has a measure of \( 2\pi \) radians:

\[ 360° = 2\pi \text{ radians.} \]

An angle of 180° is half the size of an angle of 360°, so an angle of 180° has half the radian measure of an angle of 360°.

\[ 180° = \frac{1}{2}(2\pi) \text{ radians} = \pi \text{ radians} \quad \text{Degree/radian relationship} \]

We can use the relationship \( 180° = \pi \text{ radians} \) to develop a method for converting between degrees and radians as follows.

\[ 1° = \frac{\pi}{180} \text{ radian} \quad \text{Divide by 180.} \quad \text{or} \quad 1 \text{ radian} = \frac{180°}{\pi} \quad \text{Divide by } \pi. \]

Converting Between Degrees and Radians

1. Multiply a degree measure by \( \frac{\pi}{180} \text{ radian} \) and simplify to convert to radians.
2. Multiply a radian measure by \( \frac{180°}{\pi} \) and simplify to convert to degrees.
6.1 Radian Measure

Some calculators (in radian mode) have the capability to convert directly between decimal degrees and radians. This screen shows the conversions for Example 1. Note that when exact values involving \(\pi\) are required, such as \(\frac{\pi}{4}\) in part (a), calculator approximations are not acceptable.

\[
\begin{align*}
45^\circ & \quad 0.7853981634 \\
\pi/4 & \quad 0.7853981634 \\
249.8^\circ & \quad 4.359832471
\end{align*}
\]

EXAMPLE 1 Converting Degrees to Radians

Convert each degree measure to radians.

(a) \(45^\circ\) 
(b) \(249.8^\circ\)

Solution

(a) \(45^\circ = 45 \left(\frac{\pi}{180}\right) \approx 0.785\) radian

(b) \(249.8^\circ = 249.8 \left(\frac{\pi}{180}\right) \approx 4.360\) radians

Now try Exercises 1 and 13.

EXAMPLE 2 Converting Radians to Degrees

Convert each radian measure to degrees.

(a) \(\frac{9\pi}{4}\) 
(b) \(4.25\) (Give the answer in decimal degrees.)

Solution

(a) \(\frac{9\pi}{4} = 9 \left(\frac{\pi}{4}\right) = 67.5^\circ\)

(b) \(4.25 = 4.25 \left(\frac{180^\circ}{\pi}\right) \approx 243.5^\circ\)

Now try Exercises 21 and 31.

If no unit of angle measure is specified, then radian measure is understood.

CAUTION Figure 2 shows angles measuring 30 radians and 30°. Be careful not to confuse them.

The following table and Figure 3 on the next page give some equivalent angles measured in degrees and radians. Keep in mind that \(180^\circ = \pi\) radians.

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
<th>Degrees</th>
<th>Radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>Approximate</td>
<td>Exact</td>
<td>Approximate</td>
</tr>
<tr>
<td>0°</td>
<td>0</td>
<td>90°</td>
<td>(\frac{\pi}{2})</td>
</tr>
<tr>
<td>30°</td>
<td>(\frac{\pi}{6})</td>
<td>.52</td>
<td>180°</td>
</tr>
<tr>
<td>45°</td>
<td>(\frac{\pi}{4})</td>
<td>.79</td>
<td>270°</td>
</tr>
<tr>
<td>60°</td>
<td>(\frac{\pi}{3})</td>
<td>1.05</td>
<td>360°</td>
</tr>
</tbody>
</table>
Looking Ahead to Calculus

In calculus, radian measure is much easier to work with than degree measure. If \( x \) is measured in radians, then the derivative of \( f(x) = \sin x \) is

\[
f'(x) = \cos x.
\]

However, if \( x \) is measured in degrees, then the derivative of \( f(x) = \sin x \) is

\[
f'(x) = \frac{\pi}{180} \cos x.
\]

We use radian measure to simplify certain formulas, two of which follow. Each would be more complicated if expressed in degrees.

**Arc Length of a Circle** We use the first formula to find the length of an arc of a circle. This formula is derived from the fact (proven in geometry) that the length of an arc is proportional to the measure of its central angle.

In Figure 4, angle \( QOP \) has measure 1 radian and intercepts an arc of length \( r \) on the circle. Angle \( ROT \) has measure \( \theta \) radians and intercepts an arc of length \( s \) on the circle. Since the lengths of the arcs are proportional to the measures of their central angles,

\[
\frac{s}{r} = \frac{\theta}{1}.
\]

Multiplying both sides by \( r \) gives the following result.

**Arc Length**

The length \( s \) of the arc intercepted on a circle of radius \( r \) by a central angle of measure \( \theta \) radians is given by the product of the radius and the radian measure of the angle, or

\[
s = r\theta, \quad \theta \text{ in radians.}
\]

**CAUTION** When applying the formula \( s = r\theta \), the value of \( \theta \) must be expressed in radians.
**EXAMPLE 3** Finding Arc Length Using \( s = r\theta \)

A circle has radius 18.2 cm. Find the length of the arc intercepted by a central angle having each of the following measures.

(a) \( \frac{3\pi}{8} \) radians

(b) 144°

**Solution**

(a) As shown in the figure, \( r = 18.2 \) cm and \( \theta = \frac{3\pi}{8} \).

\[
s = r\theta \quad \text{Arc length formula}
\]

\[
s = 18.2 \left( \frac{3\pi}{8} \right) \text{ cm} \quad \text{Substitute for } r \text{ and } \theta.
\]

\[
s = \frac{54.6\pi}{8} \text{ cm} = 21.4 \text{ cm}
\]

(b) The formula \( s = r\theta \) requires that \( \theta \) be measured in radians. First, convert \( \theta \) to radians by multiplying 144° by \( \frac{\pi}{180} \) radian.

\[
144^\circ = 144 \left( \frac{\pi}{180} \right) = \frac{4\pi}{5} \text{ radians} \quad \text{Convert from degrees to radians.}
\]

The length \( s \) is given by

\[
s = r\theta = 18.2 \left( \frac{4\pi}{5} \right) = \frac{72.8\pi}{5} \approx 45.7 \text{ cm.}
\]

Now try Exercises 49 and 51.

**EXAMPLE 4** Using Latitudes to Find the Distance Between Two Cities

Reno, Nevada, is approximately due north of Los Angeles. The latitude of Reno is 40° N, while that of Los Angeles is 34° N. (The N in 34° N means north of the equator.) The radius of Earth is 6400 km. Find the north-south distance between the two cities.

**Solution** Latitude gives the measure of a central angle with vertex at Earth’s center whose initial side goes through the equator and whose terminal side goes through the given location. As shown in Figure 5, the central angle between Reno and Los Angeles is \( 40^\circ - 34^\circ = 6^\circ \). The distance between the two cities can be found by the formula \( s = r\theta \); after \( 6^\circ \) is first converted to radians.

\[
6^\circ = 6 \left( \frac{\pi}{180} \right) = \frac{\pi}{30} \text{ radian}
\]

The distance between the two cities is

\[
s = r\theta = 6400 \left( \frac{\pi}{30} \right) \approx 670 \text{ km.} \quad \text{Let } r = 6400 \text{ and } \theta = \frac{\pi}{30}.
\]

Now try Exercise 55.
EXAMPLE 5 Finding a Length Using $s = r\theta$

A rope is being wound around a drum with radius .8725 ft. (See Figure 6.) How much rope will be wound around the drum if the drum is rotated through an angle of 39.72°?

Solution The length of rope wound around the drum is the arc length for a circle of radius .8725 ft and a central angle of 39.72°. Use the formula $s = r\theta$, with the angle converted to radian measure. The length of the rope wound around the drum is approximately

$$s = r\theta = .8725 \left( \frac{\pi}{180} \right) = .6049 \text{ ft.}$$

Now try Exercise 61(a).

EXAMPLE 6 Finding an Angle Measure Using $s = r\theta$

Two gears are adjusted so that the smaller gear drives the larger one, as shown in Figure 7. If the smaller gear rotates through 225°, through how many degrees will the larger gear rotate?

Solution First find the radian measure of the angle, and then find the arc length on the smaller gear that determines the motion of the larger gear. Since $225^\circ = \frac{5\pi}{4}$ radians, for the smaller gear,

$$s = r\theta = 2.5 \left( \frac{5\pi}{4} \right) = \frac{12.5\pi}{4} = \frac{25\pi}{8} \text{ cm.}$$

An arc with this length on the larger gear corresponds to an angle measure $\theta$, in radians, where

$$s = r\theta$$

$$\frac{25\pi}{8} = 4.8\theta \quad \text{Substitute} \frac{25\pi}{8} \text{ for } s \text{ and } 4.8 \text{ for } r.$$

$$\frac{125\pi}{192} = \theta. \quad \text{Substitute } \frac{25\pi}{8} \text{ as } \frac{25\pi}{4}; \text{ multiply by } \frac{1}{25} \text{ to solve for } \theta.$$

Converting $\theta$ back to degrees shows that the larger gear rotates through

$$\frac{125\pi}{192} \left( \frac{180^\circ}{\pi} \right) \approx 117^\circ. \quad \text{Convert } \theta = \frac{125\pi}{192} \text{ to degrees.}$$

Now try Exercise 63.

Area of a Sector of a Circle A sector of a circle is the portion of the interior of a circle intercepted by a central angle. Think of it as a “piece of pie.” See Figure 8. A complete circle can be thought of as an angle with measure $2\pi$ radians. If a central angle for a sector has measure $\theta$ radians, then the sector makes up the fraction $\frac{\theta}{2\pi}$ of a complete circle. The area of a complete circle with radius $r$ is $A = \pi r^2$. Therefore,

$$\text{area of the sector} = \frac{\theta}{2\pi} (\pi r^2) = \frac{1}{2} r^2 \theta, \quad \theta \text{ in radians.}$$
This discussion is summarized as follows.

**Area of a Sector**

The area of a sector of a circle of radius \( r \) and central angle \( \theta \) is given by

\[
A = \frac{1}{2} r^2 \theta, \quad \theta \text{ in radians.}
\]

**CAUTION** As in the formula for arc length, the value of \( \theta \) must be in radians when using this formula for the area of a sector.

**EXAMPLE 7** Finding the Area of a Sector-Shaped Field

Find the area of the sector-shaped field shown in Figure 9.

**Solution** First, convert \( 15^\circ \) to radians.

\[
15^\circ = 15 \left( \frac{\pi}{180} \right) = \frac{\pi}{12} \text{ radian}
\]

Now use the formula to find the area of a sector of a circle with radius \( r = 321 \).

\[
A = \frac{1}{2} r^2 \theta = \frac{1}{2} (321)^2 \left( \frac{\pi}{12} \right) \approx 13,500 \text{ m}^2
\]

Now try Exercise 77.

**6.1 Exercises**

1. \( \frac{\pi}{3} \)  
2. \( \frac{\pi}{2} \)  
3. \( \frac{5\pi}{6} \)  
4. \( \frac{3\pi}{2} \)  
5. \( -\frac{7\pi}{6} \)  
6. \( \frac{8\pi}{3} \)  
7. \( -\frac{\pi}{4} \)  
8. \( -\frac{7\pi}{6} \)  
9. \( .68 \)  
10. \( 1.29 \)  
11. \( 2.43 \)  
12. \( 3.05 \)  
13. \( 1.122 \)  
14. \( 2.140 \)  
15. \( 1 \)  
16. \( 2 \)  

**Convert each degree measure to radians. Leave answers as multiples of \( \pi \). See Example 1(a).**

1. \( 60^\circ \)  
2. \( 90^\circ \)  
3. \( 150^\circ \)  
4. \( 270^\circ \)  
5. \( 315^\circ \)  
6. \( 480^\circ \)  
7. \( -45^\circ \)  
8. \( -210^\circ \)  

**Convert each degree measure to radians. See Example 1(b).**

9. \( 39^\circ \)  
10. \( 74^\circ \)  
11. \( 139^\circ 10' \)  
12. \( 174^\circ 50' \)  
13. \( 64.29^\circ \)  
14. \( 122.62^\circ \)  

**Concept Check** In Exercises 15–18, each angle \( \theta \) is an integer when measured in radians. Give the radian measure of the angle.

15.  
16.
In your own words, explain how to convert
(a) degree measure to radian measure; (b) radian measure to degree measure.

Explain the difference between degree measure and radian measure.

Convert each radian measure to degrees. See Example 2(a).

21. \(\frac{\pi}{3}\) 22. \(\frac{8\pi}{3}\) 23. \(\frac{7\pi}{4}\) 24. \(\frac{2\pi}{3}\)
25. \(\frac{11\pi}{6}\) 26. \(\frac{15\pi}{4}\) 27. \(-\frac{\pi}{6}\) 28. \(-\frac{7\pi}{20}\)

Convert each radian measure to degrees. Give answers using decimal degrees to the nearest tenth. See Example 2(b).

29. 2 30. 5 31. 1.74
32. 0.3417 33. -9.84763 34. -3.47189

Relating Concepts
For individual or collaborative investigation
(Exercises 35–42)

In anticipation of the material in the next section, we show how to find the trigonometric function values of radian-measured angles. Suppose we want to find \(\sin \frac{5\pi}{6}\). One way to do this is to convert \(\frac{5\pi}{6}\) radians to 150°, and then use the methods of Chapter 5 to evaluate:

\[
\sin \frac{5\pi}{6} = \sin 150° = + \sin 30° = \frac{1}{2}. \quad \text{(Section 5.3)}
\]

Sine is positive in quadrant II. Reference angle for 150°

Use this technique to find each function value. Give exact values.

35. \(\tan \frac{\pi}{4}\) 36. \(\csc \frac{\pi}{4}\) 37. \(\cot \frac{2\pi}{3}\) 38. \(\cos \frac{\pi}{3}\)
39. \(\sec \pi\) 40. \(\sin \left(-\frac{7\pi}{6}\right)\) 41. \(\cos \left(-\frac{\pi}{6}\right)\) 42. \(\tan \left(-\frac{9\pi}{4}\right)\)
43. $2\pi$  44. $4\pi$  45. 8  46. 6  
47. 1  48. 1.5  49. 25.8 cm  
50. 3.08 cm  51. 5.05 m  52. 169 cm  53. The length is doubled.  
54. $s = \frac{\pi r \theta}{180}$  
55. 3500 km  56. 1500 km  57. 5900 km  58. 8800 km

Concept Check  Find the exact length of each arc intercepted by the given central angle.

43. 

44. 

Concept Check  Find the radius of each circle.

45. 

46. 

Concept Check  Find the measure of each central angle (in radians).

47. 

48. 

Unless otherwise directed, give calculator approximations in your answers in the rest of this exercise set.

Find the length of each arc intercepted by a central angle $\theta$ in a circle of radius $r$. See Example 3.

49. $r = 12.3$ cm, $\theta = \frac{2\pi}{3}$ radians  
50. $r = .892$ cm, $\theta = \frac{11\pi}{10}$ radians  
51. $r = 4.82$ m, $\theta = 60^\circ$  
52. $r = 71.9$ cm, $\theta = 135^\circ$  
53. Concept Check  If the radius of a circle is doubled, how is the length of the arc intercepted by a fixed central angle changed?

54. Concept Check  Radian measure simplifies many formulas, such as the formula for arc length, $s = r\theta$. Give the corresponding formula when $\theta$ is measured in degrees instead of radians.

Distance Between Cities  Find the distance in kilometers between each pair of cities, assuming they lie on the same north-south line. See Example 4.

55. Panama City, Panama, 9° N, and Pittsburgh, Pennsylvania, 40° N
56. Farmersville, California, 36° N, and Penticton, British Columbia, 49° N
57. New York City, New York, 41° N, and Lima, Peru, 12° S
58. Halifax, Nova Scotia, 45° N, and Buenos Aires, Argentina, 34° S
542 CHAPTER 6 The Circular Functions and Their Graphs

59. Latitude of Madison Madison, South Dakota, and Dallas, Texas, are 1200 km apart and lie on the same north-south line. The latitude of Dallas is 33° N. What is the latitude of Madison?

60. Latitude of Toronto Charleston, South Carolina, and Toronto, Canada, are 1100 km apart and lie on the same north-south line. The latitude of Charleston is 33° N. What is the latitude of Toronto?

Work each problem. See Examples 5 and 6.

61. Pulley Raising a Weight

(a) How many inches will the weight in the figure rise if the pulley is rotated through an angle of 71° 50′?
(b) Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 6 in.?

62. Pulley Raising a Weight Find the radius of the pulley in the figure if a rotation of 51.6° raises the weight 11.4 cm.

63. Rotating Wheels The rotation of the smaller wheel in the figure causes the larger wheel to rotate. Through how many degrees will the larger wheel rotate if the smaller one rotates through 60.0°?

64. Rotating Wheels Find the radius of the larger wheel in the figure if the smaller wheel rotates 80.0° when the larger wheel rotates 50.0°.

65. Bicycle Chain Drive The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through 180°? Assume the radius of the bicycle wheel is 13.6 in.

66. Pickup Truck Speedometer The speedometer of Terry’s small pickup truck is designed to be accurate with tires of radius 14 in.

(a) Find the number of rotations of a tire in 1 hr if the truck is driven at 55 mph.
(b) Suppose that oversize tires of radius 16 in. are placed on the truck. If the truck is now driven for 1 hr with the speedometer reading 55 mph, how far has the truck gone? If the speed limit is 55 mph, does Terry deserve a speeding ticket?

If a central angle is very small, there is little difference in length between an arc and the inscribed chord. See the figure. Approximate each of the following lengths by finding the necessary arc length. (Note: When a central angle intercepts an arc, the arc is said to subtend the angle.)

67. **Length of a Train** A railroad track in the desert is 3.5 km away. A train on the track subtends (horizontally) an angle of 2° 10'. Find the length of the train.

68. **Distance to a Boat** The mast of Brent Simon’s boat is 32 ft high. If it subtends an angle of 5°, how far away is it?

Concept Check Find the area of each sector.

69. \[ \text{Area} = \theta \cdot \frac{r^2}{2} \]

70. \[ \text{Area} = \theta \cdot \frac{r^2}{2} \]

Concept Check Find the measure (in radians) of each central angle. The number inside the sector is the area.

71. \[ \text{Area} = 3 \text{ sq units} \]

72. \[ \text{Area} = 8 \text{ sq units} \]

Find the area of a sector of a circle having radius \( r \) and central angle \( \theta \). See Example 7.

73. \( r = 29.2 \text{ m}, \theta = \frac{5\pi}{6} \text{ radians} \)

74. \( r = 59.8 \text{ km}, \theta = \frac{2\pi}{3} \text{ radians} \)

75. \( r = 30.0 \text{ ft}, \theta = \frac{\pi}{2} \text{ radians} \)

76. \( r = 90.0 \text{ yd}, \theta = \frac{5\pi}{6} \text{ radians} \)

77. \( r = 12.7 \text{ cm}, \theta = 81^\circ \)

78. \( r = 18.3 \text{ m}, \theta = 125^\circ \)

79. \( r = 40.0 \text{ mi}, \theta = 135^\circ \)

80. \( r = 90.0 \text{ km}, \theta = 270^\circ \)

Work each problem.

81. Find the measure (in radians) of a central angle of a sector of area 16 \text{ in.}^2 in a circle of radius 3.0 in.

82. Find the radius of a circle in which a central angle of \( \frac{\pi}{5} \text{ radian} \) determines a sector of area 64 \text{ m}^2.
83. **Measures of a Structure**  The figure shows Medicine Wheel, a Native American structure in northern Wyoming. This circular structure is perhaps 2500 yr old. There are 27 aboriginal spokes in the wheel, all equally spaced.

(a) Find the measure of each central angle in degrees and in radians.
(b) If the radius of the wheel is 76 ft, find the circumference.
(c) Find the length of each arc intercepted by consecutive pairs of spokes.
(d) Find the area of each sector formed by consecutive spokes.

84. **Area Cleaned by a Windshield Wiper**  The Ford Model A, built from 1928 to 1931, had a single windshield wiper on the driver’s side. The total arm and blade was 10 in. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-in. wiper blade. What is the area of the region cleaned?

85. **Circular Railroad Curves**  In the United States, circular railroad curves are designated by the degree of curvature, the central angle subtended by a chord of 100 ft. Suppose a portion of track has curvature 42°. (Source: Hay, W., *Railroad Engineering*, John Wiley & Sons, 1982.)

(a) What is the radius of the curve?
(b) What is the length of the arc determined by the 100-ft chord?
(c) What is the area of the portion of the circle bounded by the arc and the 100-ft chord?

86. **Land Required for a Solar-Power Plant**  A 300-megawatt solar-power plant requires approximately 950,000 m² of land area in order to collect the required amount of energy from sunlight.

(a) If this land area is circular, what is its radius?
(b) If this land area is a 35° sector of a circle, what is its radius?

87. **Area of a Lot**  A frequent problem in surveying city lots and rural lands adjacent to curves of highways and railways is that of finding the area when one or more of the boundary lines is the arc of a circle. Find the area of the lot shown in the figure. (Source: Anderson, J. and E. Michael, *Introduction to Surveying*, McGraw-Hill, 1985.)

88. **Nautical Miles**  Nautical miles are used by ships and airplanes. They are different from statute miles, which equal 5280 ft. A nautical mile is defined to be the arc length along the equator intercepted by a central angle AOB of 1 min, as illustrated in the figure. If the equatorial radius of Earth is 3963 mi, use the arc length formula to approximate the number of statute miles in 1 nautical mile. Round your answer to two decimal places.
89. Circumference of Earth  The first accurate estimate of the distance around Earth was done by the Greek astronomer Eratosthenes (276–195 B.C.), who noted that the noontime position of the sun at the summer solstice differed by 7° 12’ from the city of Syene to the city of Alexandria. (See the figure.) The distance between these two cities is 496 mi. Use the arc length formula to estimate the radius of Earth. Then find the circumference of Earth. 


90. Diameter of the Moon  The distance to the moon is approximately 238,900 mi. Use the arc length formula to estimate the diameter d of the moon if angle θ in the figure is measured to be .517°.

91. Concept Check  If the radius of a circle is doubled and the central angle of a sector is unchanged, how is the area of the sector changed?

92. Concept Check  Give the corresponding formula for the area of a sector when the angle is measured in degrees.

6.2 The Unit Circle and Circular Functions  

Circular Functions  •  Finding Values of Circular Functions  •  Determining a Number with a Given Circular Function Value  •  Angular and Linear Speed

In Section 5.2, we defined the six trigonometric functions in such a way that the domain of each function was a set of angles in standard position. These angles can be measured in degrees or in radians. In advanced courses, such as calculus, it is necessary to modify the trigonometric functions so that their domains consist of real numbers rather than angles. We do this by using the relationship between an angle θ and an arc of length s on a circle.

Circular Functions  In Figure 10, we start at the point (1, 0) and measure an arc of length s along the circle. If s > 0, then the arc is measured in a counterclockwise direction, and if s < 0, then the direction is clockwise. (If s = 0, then no arc is measured.) Let the endpoint of this arc be at the point (x, y). The circle in Figure 10 is a unit circle—it has center at the origin and radius 1 unit (hence the name unit circle). Recall from algebra that the equation of this circle is

$$x^2 + y^2 = 1.$$  (Section 2.1)
We saw in the previous section that the radian measure of $\theta$ is related to the arc length $s$. In fact, for $\theta$ measured in radians, we know that $s = r\theta$. Here, $r = 1$, so $s$, which is measured in linear units such as inches or centimeters, is numerically equal to $\theta$, measured in radians. Thus, the trigonometric functions of angle $\theta$ in radians found by choosing a point $(x, y)$ on the unit circle can be rewritten as functions of the arc length $s$, a real number. When interpreted this way, they are called **circular functions**.

### Circular Functions

$$
\sin s = y \quad \cos s = x \quad \tan s = \frac{y}{x} \quad (x \neq 0)
$$

$$
\csc s = \frac{1}{y} \quad (y \neq 0) \quad \sec s = \frac{1}{x} \quad (x \neq 0) \quad \cot s = \frac{x}{y} \quad (y \neq 0)
$$

Since $x$ represents the cosine of $s$ and $y$ represents the sine of $s$, and because of the discussion in Section 6.1 on converting between degrees and radians, we can summarize a great deal of information in a concise manner, as seen in Figure 11.\(^*\)

\[\text{Unit circle } x^2 + y^2 = 1\]

**Figure 11**

**NOTE** Since $\sin s = y$ and $\cos s = x$, we can replace $x$ and $y$ in the equation $x^2 + y^2 = 1$ and obtain the Pythagorean identity

$$\cos^2 s + \sin^2 s = 1.$$

The ordered pair $(x, y)$ represents a point on the unit circle, and therefore

$$-1 \leq x \leq 1 \quad \text{and} \quad -1 \leq y \leq 1,$$

so

$$-1 \leq \cos s \leq 1 \quad \text{and} \quad -1 \leq \sin s \leq 1.$$

\[^*\text{The authors thank Professor Marvel Townsend of the University of Florida for her suggestion to include this figure.}\]
For any value of \( s \), both \( \sin s \) and \( \cos s \) exist, so the domain of these functions is the set of all real numbers. For \( \tan s \), defined as \( \frac{\sin s}{\cos s} \), \( x \) must not equal 0. The only way \( x \) can equal 0 is when the arc length \( s \) is \( \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2} \), and so on. To avoid a 0 denominator, the domain of the tangent function must be restricted to those values of \( s \) satisfying

\[
 s \neq (2n + 1) \frac{\pi}{2}, \quad n \text{ any integer.}
\]

The definition of secant also has \( x \) in the denominator, so the domain of secant is the same as the domain of tangent. Both cotangent and cosecant are defined with a denominator of \( y \). To guarantee that \( y \neq 0 \), the domain of these functions must be the set of all values of \( s \) satisfying

\[
 s \neq n\pi, \quad n \text{ any integer.}
\]

In summary, the domains of the circular functions are as follows.

### Domains of the Circular Functions

Assume that \( n \) is any integer and \( s \) is a real number.

**Sine and Cosine Functions:** \((-\infty, \infty)\)

**Tangent and Secant Functions:** \( \left\{ s \mid s \neq (2n + 1) \frac{\pi}{2} \right\} \)

**Cotangent and Cosecant Functions:** \( \{ s \mid s \neq n\pi \}\)

### Finding Values of Circular Functions

The circular functions (functions of real numbers) are closely related to the trigonometric functions of angles measured in radians. To see this, let us assume that angle \( \theta \) is in standard position, superimposed on the unit circle, as shown in Figure 12. Suppose further that \( \theta \) is the radian measure of this angle. Using the arc length formula \( s = r\theta \) with \( r = 1 \), we have \( s = \theta \). Thus, the length of the intercepted arc is the real number that corresponds to the radian measure of \( \theta \). Using the definitions of the trigonometric functions, we have

\[
 \sin \theta = \frac{y}{r} = \frac{y}{1} = y = \sin s, \quad \text{and} \quad \cos \theta = \frac{x}{r} = \frac{x}{1} = x = \cos s,
\]

and so on. As shown here, the trigonometric functions and the circular functions lead to the same function values, provided we think of the angles as being in radian measure. This leads to the following important result concerning evaluation of circular functions.

### Evaluating a Circular Function

Circular function values of real numbers are obtained in the same manner as trigonometric function values of angles measured in radians. This applies both to methods of finding exact values (such as reference angle analysis) and to calculator approximations. Calculators must be in radian mode when finding circular function values.
EXAMPLE 1 Finding Exact Circular Function Values

Find the exact values of \(\sin \frac{3\pi}{2}\), \(\cos \frac{3\pi}{2}\), and \(\tan \frac{3\pi}{2}\).

Solution Evaluating a circular function at the real number \(\frac{3\pi}{2}\) is equivalent to evaluating it at \(\frac{3\pi}{2}\) radians. An angle of \(\frac{3\pi}{2}\) radians intersects the unit circle at the point \((0, -1)\), as shown in Figure 13. Since

\[
\sin s = y, \quad \cos s = x, \quad \text{and} \quad \tan s = \frac{y}{x},
\]

it follows that

\[
\sin \frac{3\pi}{2} = -1, \quad \cos \frac{3\pi}{2} = 0, \quad \text{and} \quad \tan \frac{3\pi}{2} \text{ is undefined.}
\]

Now try Exercise 1.

EXAMPLE 2 Finding Exact Circular Function Values

(a) Use Figure 11 to find the exact values of \(\cos \frac{7\pi}{4}\) and \(\sin \frac{7\pi}{4}\).

(b) Use Figure 11 to find the exact value of \(\tan \left( -\frac{5\pi}{4} \right)\).

(c) Use reference angles and degree/radian conversion to find the exact value of \(\cos \frac{2\pi}{3}\).

Solution

(a) In Figure 11, we see that the terminal side of \(\frac{7\pi}{4}\) radians intersects the unit circle at \(\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\). Thus,

\[
\cos \frac{7\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2}.
\]

(b) Angles of \(-\frac{5\pi}{4}\) radians and \(\frac{7\pi}{4}\) radians are coterminal. Their terminal sides intersect the unit circle at \(\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\), so

\[
\tan \left( -\frac{5\pi}{3} \right) = \tan \frac{\pi}{3} = \frac{\sqrt{3}}{3} = \sqrt{3}.
\]

(c) An angle of \(\frac{2\pi}{3}\) radians corresponds to an angle of 120°. In standard position, 120° lies in quadrant II with a reference angle of 60°, so

\[
\cos \frac{2\pi}{3} = \cos 120° = -\cos 60° = -\frac{1}{2}.
\]

Now try Exercises 7, 17, and 21.

NOTE Examples 1 and 2 illustrate that there are several methods of finding exact circular function values.
6.2 The Unit Circle and Circular Functions

**EXAMPLE 3** Approximating Circular Function Values

Find a calculator approximation to four decimal places for each circular function value.

(a) \( \cos 1.85 \)  
(b) \( \cos .5149 \)  
(c) \( \cot 1.3209 \)  
(d) \( \sec(-2.9234) \)

**Solution**

(a) With a calculator in radian mode, we find \( \cos 1.85 \approx -0.2756 \).

(b) \( \cos .5149 \approx 0.8703 \)  
   Use a calculator in radian mode.

(c) As before, to find cotangent, secant, and cosecant function values, we
   must use the appropriate reciprocal functions. To find \( \cot 1.3209 \), first find
   \( \tan 1.3209 \) and then find the reciprocal.
   \[
   \cot 1.3209 = \frac{1}{\tan 1.3209} \approx 0.2552
   \]

(d) \( \sec(-2.9234) = \frac{1}{\cos(-2.9234)} \approx -1.0243 \)

Now try Exercises 23, 29, and 33.

**CAUTION** A common error in trigonometry is using calculators in degree mode when radian mode should be used. Remember, *if you are finding a circular function value of a real number, the calculator must be in radian mode.*

**Determining a Number with a Given Circular Function Value** Recall from Section 5.3 how we used a calculator to determine an angle measure, given a trigonometric function value of the angle.

**EXAMPLE 4** Finding a Number Given Its Circular Function Value

(a) Approximate the value of \( s \) in the interval \( [0, \pi] \), if \( \cos s = .9685 \).

(b) Find the exact value of \( s \) in the interval \( [\pi, \frac{3\pi}{2}] \), if \( \tan s = 1 \).

**Solution**

(a) Since we are given a cosine value and want to determine the real number in
   \( [0, \pi] \) having this cosine value, we use the inverse cosine function of a
   calculator. With the calculator in radian mode, we find
   \[
   \cos^{-1}(0.9685) \approx 0.2517. \quad \text{(Section 5.3)}
   \]
   See Figure 14. (Refer to your owner’s manual to determine how to evaluate
   the \( \sin^{-1} \), \( \cos^{-1} \), and \( \tan^{-1} \) functions with your calculator.)

(b) Recall that \( \tan \frac{\pi}{4} = 1 \), and in quadrant III \( \tan s \) is positive. Therefore,
   \[
   \tan \left( \pi + \frac{\pi}{4} \right) = \tan \frac{5\pi}{4} = 1,
   \]
   and \( s = \frac{5\pi}{4} \). Figure 15 supports this result.

Now try Exercises 49 and 55.
A convenient way to see the sine, cosine, and tangent trigonometric ratios geometrically is shown in Figure 16 for $\theta$ in quadrants I and II. The circle shown is the unit circle, which has radius 1. By remembering this figure and the segments that represent the sine, cosine, and tangent functions, you can quickly recall properties of the trigonometric functions. Horizontal line segments to the left of the origin and vertical line segments below the $x$-axis represent negative values. Note that the tangent line must be tangent to the circle at $(1, 0)$, for any quadrant in which $\theta$ lies.

\[ PQ = y = \frac{y}{1} = \sin \theta; \]
\[ OQ = x = \frac{x}{1} = \cos \theta; \]
\[ AB = \frac{AB}{1} = \frac{AB}{AO} = \frac{y}{x} \]
(by similar triangles) $= \tan \theta$

**Angular and Linear Speed** The human joint that can be flexed the fastest is the wrist, which can rotate through $90^\circ$, or $\frac{\pi}{2}$ radians, in .045 sec while holding a tennis racket. **Angular speed** $\omega$ (omega) measures the speed of rotation and is defined by

\[ \omega = \frac{\theta}{t}, \]

where $\theta$ is the angle of rotation in radians and $t$ is time. The angular speed of a human wrist swinging a tennis racket is

\[ \omega = \frac{\theta}{t} = \frac{\frac{\pi}{2}}{.045} \approx 35 \text{ radians per sec}. \]

The **linear speed** $v$ at which the tip of the racket travels as a result of flexing the wrist is given by

\[ v = r\omega, \]

where $r$ is the radius (distance) from the tip of the racket to the wrist joint. If $r = 2$ ft, then the speed at the tip of the racket is

\[ v = r\omega = 2(35) = 70 \text{ ft per sec, or about 48 mph.} \]
In a tennis serve the arm rotates at the shoulder, so the final speed of the racket is considerably faster. (*Source: Cooper, J. and R. Glassow, *Kinesiology*, Second Edition, C.V. Mosby, 1968.)*

**EXAMPLE 5** Finding Angular Speed of a Pulley and Linear Speed of a Belt

A belt runs a pulley of radius 6 cm at 80 revolutions per min.

(a) Find the angular speed of the pulley in radians per second.

(b) Find the linear speed of the belt in centimeters per second.

**Solution**

(a) In 1 min, the pulley makes 80 revolutions. Each revolution is \(2\pi\) radians, for a total of

\[
80(2\pi) = 160\pi \text{ radians per min.}
\]

Since there are 60 sec in 1 min, we find \(\omega\), the angular speed in radians per second, by dividing 160\(\pi\) by 60.

\[
\omega = \frac{160\pi}{60} = \frac{8\pi}{3} \text{ radians per sec}
\]

(b) The linear speed of the belt will be the same as that of a point on the circumference of the pulley. Thus,

\[
v = r\omega = 6\left(\frac{8\pi}{3}\right) = 16\pi \approx 50.3 \text{ cm per sec.}
\]

Now try Exercise 95.

Suppose that an object is moving at a constant speed. If it travels a distance \(s\) in time \(t\), then its linear speed \(v\) is given by

\[
v = \frac{s}{t}.
\]

If the object is traveling in a circle, then \(s = r\theta\), where \(r\) is the radius of the circle and \(\theta\) is the angle of rotation. Thus, we can write

\[
v = \frac{r\theta}{t}.
\]

The formulas for angular and linear speed are summarized in the table.

<table>
<thead>
<tr>
<th>Angular Speed</th>
<th>Linear Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega = \frac{\theta}{t})</td>
<td>(v = \frac{s}{t})</td>
</tr>
<tr>
<td>((\omega) in radians per unit time, (\theta) in radians)</td>
<td>(v = \frac{r\theta}{t})</td>
</tr>
<tr>
<td></td>
<td>(v = r\omega)</td>
</tr>
</tbody>
</table>
EXAMPLE 6  Finding Linear Speed and Distance Traveled by a Satellite

A satellite traveling in a circular orbit 1600 km above the surface of Earth takes 2 hr to make an orbit. The radius of Earth is 6400 km. See Figure 18.

(a) Find the linear speed of the satellite.

(b) Find the distance the satellite travels in 4.5 hr.

Solution

(a) The distance of the satellite from the center of Earth is

\[ r = 1600 + 6400 = 8000 \text{ km}. \]

For one orbit, \( \theta = 2\pi \), and

\[ s = 8000(2\pi) \text{ km}. \]  \hspace{1cm} (Section 6.1)

Since it takes 2 hr to complete an orbit, the linear speed is

\[ v = \frac{s}{t} = \frac{8000(2\pi)}{2} = 8000\pi \approx 25,000 \text{ km per hr}. \]

(b) \[ s = vt = 8000\pi(4.5) = 36,000\pi = 110,000 \text{ km} \]

Now try Exercise 93.

6.2 Exercises

For each value of \( \theta \), find (a) \( \sin \theta \), (b) \( \cos \theta \), and (c) \( \tan \theta \). See Example 1.

1. \( \theta = \frac{\pi}{2} \)  
2. \( \theta = \pi \)  
3. \( \theta = 2\pi \)  
4. \( \theta = 3\pi \)  
5. \( \theta = -\pi \)  
6. \( \theta = -\frac{3\pi}{2} \)

Find the exact circular function value for each of the following. See Example 2.

7. \( \sin \frac{7\pi}{6} \)  
8. \( \cos \frac{5\pi}{3} \)  
9. \( \tan \frac{3\pi}{4} \)  
10. \( \sec \frac{2\pi}{3} \)  
11. \( \csc \frac{11\pi}{6} \)  
12. \( \cot \frac{5\pi}{6} \)  
13. \( \cos \left(-\frac{4\pi}{3}\right) \)  
14. \( \tan \frac{17\pi}{3} \)  
15. \( \cos \frac{7\pi}{4} \)  
16. \( \sec \frac{5\pi}{4} \)  
17. \( \sin \left(-\frac{4\pi}{3}\right) \)  
18. \( \sin \left(-\frac{5\pi}{6}\right) \)  
19. \( \sec \frac{23\pi}{6} \)  
20. \( \csc \frac{13\pi}{3} \)  
21. \( \tan \frac{5\pi}{6} \)  
22. \( \cos \frac{3\pi}{4} \)

Find a calculator approximation for each circular function value. See Example 3.

23. \( \sin .6109 \)  
24. \( \sin .8203 \)  
25. \( \cos(-1.1519) \)  
26. \( \cos(-5.2825) \)  
27. \( \tan 4.0203 \)  
28. \( \tan 6.4752 \)  
29. \( \csc(-9.4946) \)  
30. \( \csc 1.3875 \)  
31. \( \sec 2.8440 \)  
32. \( \sec(-8.3429) \)  
33. \( \cot 6.0301 \)  
34. \( \cot 3.8426 \)
6.2 The Unit Circle and Circular Functions

35. 36. –.75 37. 4
38. 44 39. negative
40. negative 41. negative
42. positive 43. positive
44. negative

45. \( \sin \theta = \frac{\sqrt{2}}{2} \); \( \cos \theta = \frac{\sqrt{2}}{2} \);
\( \tan \theta = 1; \cot \theta = 1; \)
\( \sec \theta = \sqrt{2}; \csc \theta = \sqrt{2} \)
46. \( \sin \theta = \frac{8}{17}; \cos \theta = -\frac{15}{17} \);
\( \tan \theta = -\frac{8}{15}; \cot \theta = -\frac{15}{8}; \)
\( \sec \theta = -\frac{17}{15}; \csc \theta = \frac{17}{8} \)
47. \( \sin \theta = -\frac{12}{13}; \cos \theta = \frac{5}{13} \);
\( \tan \theta = -\frac{12}{5}; \cot \theta = -\frac{5}{12}; \)
\( \sec \theta = \frac{13}{5}; \csc \theta = -\frac{13}{12} \)
48. \( \sin \theta = -\frac{1}{2}; \cos \theta = -\frac{\sqrt{3}}{2} \);
\( \tan \theta = \frac{\sqrt{3}}{3}; \cot \theta = \sqrt{3}; \)
\( \sec \theta = -\frac{2\sqrt{3}}{3}; \csc \theta = -2 \)

49. \(.2095 \quad 50. \.6720 \quad 51. \.4426 \quad 52. \.2799 \quad 53. \.3887 \quad 54. \.3634 \quad 55. \frac{5\pi}{6} \quad 56. \frac{2\pi}{3} \)

Concept Check The figure displays a unit circle and an angle of 1 radian. The tick marks on the circle are spaced at every two-tenths radian. Use the figure to estimate each value.

35. \( \cos .8 \)
36. \( \sin 4 \)
37. an angle whose cosine is –.65
38. an angle whose sine is –.95

Concept Check Without using a calculator, decide whether each function value is positive or negative. (Hint: Consider the radian measures of the quadrant angles.)

39. \( \cos 2 \)
40. \( \sin(-1) \)
41. \( \sin 5 \)
42. \( \cos 6 \)
43. \( \tan 6.29 \)
44. \( \tan(-6.29) \)

Concept Check Each figure in Exercises 45–48 shows angle \( \theta \) in standard position with its terminal side intersecting the unit circle. Evaluate the six circular function values of \( \theta \).

45.

46. \( \frac{\sqrt{3}}{2} \quad \frac{\sqrt{2}}{2} \)

47.

48. \( -\frac{\sqrt{3}}{2} \quad -\frac{1}{2} \)

Find the value of \( s \) in the interval \([0, \frac{\pi}{2}]\) that makes each statement true. See Example 4(a).

49. \( \tan s = .2126 \)
50. \( \cos s = .7826 \)
51. \( \sin s = .9918 \)
52. \( \cot s = .2994 \)
53. \( \sec s = 1.0806 \)
54. \( \csc s = 1.0219 \)

Find the exact value of \( s \) in the given interval that has the given circular function value. Do not use a calculator. See Example 4(b).

55. \( \left[ \frac{\pi}{2}, \pi \right]; \sin s = \frac{1}{2} \)
56. \( \left[ \frac{\pi}{2}, \pi \right]; \cos s = -\frac{1}{2} \)
57. \( \frac{4\pi}{3} \)  
58. \( \frac{7\pi}{6} \)  
59. \( \frac{7\pi}{4} \)  

60. \( \frac{11\pi}{6} \)  
61. \((-0.8011, 0.5985)\)  
62. \((-0.9668, -0.2555)\)  
63. \((0.4385, -0.8987)\)  
64. \((-0.7259, 0.6878)\)  

65. 66. IV  
67. II  
68. III  
69. \( \frac{3\pi}{2} \) radian per sec  
70. \( \frac{\pi}{25} \) radian per sec  
71. \( \frac{6}{5} \) min  
72. 9 min  
73. 1.80311 radian per sec  
74. 10.768 radians  
75. \( \frac{9}{5} \) radians per sec  
76. 6 radians per sec  
77. 1.83333 radians per sec  
78. 9.29755 cm per sec  
79. 18\pi \) cm  
80. \( \frac{216\pi}{5} \) yd  
81. 12 sec  
82. \( \frac{3\pi}{32} \) radian per sec  
83. \( \frac{\pi}{6} \) radian per hr  
84. 600\pi radians per min  
85. \( \frac{7\pi}{30} \) cm per min

57. \[ \left[ \frac{3\pi}{2} \right] \text{; tan } s = \sqrt{3} \]  
58. \[ \left[ \frac{3\pi}{2} \right] \text{; sin } s = -\frac{1}{2} \]  
59. \[ \left[ \frac{3\pi}{2}, 2\pi \right] \text{; tan } s = -1 \]  
60. \[ \left[ \frac{3\pi}{2}, 2\pi \right] \text{; cos } s = \sqrt{3} \]  

Suppose an arc of length \( s \) lies on the unit circle \( x^2 + y^2 = 1 \), starting at the point \((1, 0)\) and terminating at the point \((x, y)\). (See Figure 10.) Use a calculator to find the approximate coordinates for \((x, y)\). (Hint: \( x = \cos s \) and \( y = \sin s \).)  

61. \( s = 2.5 \)  
62. \( s = 3.4 \)  
63. \( s = -7.4 \)  
64. \( s = -3.9 \)  

Concept Check  
For each value of \( s \), use a calculator to find \( \sin s \) and \( \cos s \) and then use the results to decide in which quadrant an angle of \( s \) radians lies.  

65. \( s = 51 \)  
66. \( s = 49 \)  
67. \( s = 65 \)  
68. \( s = 79 \)

Use the formula \( \omega = \frac{\theta}{t} \) to find the value of the missing variable.  

69. \( \theta = \frac{3\pi}{4} \) radians, \( t = 8 \) sec  
70. \( \theta = \frac{2\pi}{5} \) radians, \( t = 10 \) sec  

71. \( \theta = \frac{2\pi}{9} \) radian, \( \omega = \frac{5\pi}{27} \) radian per min  
72. \( \theta = \frac{3\pi}{8} \) radians, \( \omega = \frac{\pi}{24} \) radian per min  
73. \( \theta = 3.871142 \) radians, \( t = 21.4693 \) sec  
74. \( \omega = 0.90674 \) radian per min, \( t = 11.876 \) min

Use the formula \( v = r\omega \) to find the value of the missing variable.  

75. \( v = 9 \) m per sec, \( r = 5 \) m  
76. \( v = 18 \) ft per sec, \( r = 3 \) ft  
77. \( v = 107.692 \) m per sec, \( r = 58.7413 \) m  
78. \( r = 24.93215 \) cm, \( \omega = 0.372914 \) radian per sec

The formula \( \omega = \frac{\theta}{t} \) can be rewritten as \( \theta = \omega t \). Using \( \omega t \) for \( \theta \) changes \( s = r\theta \) to \( s = r\omega t \). Use the formula \( s = r\omega t \) to find the value of the missing variable.  

79. \( r = 6 \) cm, \( \omega = \frac{\pi}{3} \) radians per sec, \( t = 9 \) sec  
80. \( r = 9 \) yd, \( \omega = \frac{2\pi}{5} \) radians per sec, \( t = 12 \) sec  
81. \( s = 6\pi \) cm, \( r = 2 \) cm, \( \omega = \frac{\pi}{4} \) radian per sec  
82. \( s = \frac{3\pi}{4} \) km, \( r = 2 \) km, \( t = 4 \) sec

Find \( \omega \) for each of the following.  

83. the hour hand of a clock  
84. a line from the center to the edge of a CD revolving 300 times per min

Find \( v \) for each of the following.  

85. the tip of the minute hand of a clock, if the hand is 7 cm long
86. $1260\pi \text{ cm per min}$
87. $1500\pi \text{ m per min}$
88. $112,880\pi \text{ cm per min}$
89. $2\pi \text{ sec}$
91. 15.5 mph  
92. 24.62 hr
93. (a) $\frac{2\pi}{365}$ radian
(b) $\frac{\pi}{1380}$ radian per hr
(c) 66,700 mph
94. (a) $2\pi$ radians per day;
(b) $0$ radian per hr
(c) 12,800$\pi$ km per day or about 533 km per hr
(d) about 28,000 km per day or about 1200 km per hr
95. (a) .24 radian per sec
(b) 3.11 cm per sec
96. larger pulley: $\frac{25\pi}{18}$ radians per sec;
smaller pulley: $\frac{125\pi}{48}$ radians per sec

86. A point on the tread of a tire of radius 18 cm, rotating 35 times per min
87. The tip of an airplane propeller 3 m long, rotating 500 times per min (Hint: $r = 1.5$ m.)
88. A point on the edge of a gyroscope of radius 83 cm, rotating 680 times per min
89. **Concept Check** If a point moves around the circumference of the unit circle at the speed of 1 unit per sec, how long will it take for the point to move around the entire circle?
90. What is the difference between linear velocity and angular velocity?

Solve each problem. See Examples 5 and 6.

91. **Speed of a Bicycle** The tires of a bicycle have radius 13 in. and are turning at the rate of 200 revolutions per min. See the figure. How fast is the bicycle traveling in miles per hour? (Hint: 5280 ft = 1 mi.)

92. **Hours in a Martian Day** Mars rotates on its axis at the rate of about .2552 radian per hr. Approximately how many hours are in a Martian day? (Source: Wright, John W. (General Editor), *The Universal Almanac*, Andrews and McMeel, 1997.)

93. **Angular and Linear Speeds of Earth** Earth travels about the sun in an orbit that is almost circular. Assume that the orbit is a circle with radius 93,000,000 mi. Its angular and linear speeds are used in designing solar-power facilities.
   (a) Assume that a year is 365 days, and find the angle formed by Earth’s movement in one day.
   (b) Give the angular speed in radians per hour.
   (c) Find the linear speed of Earth in miles per hour.

94. **Angular and Linear Speeds of Earth** Earth revolves on its axis once every 24 hr. Assuming that Earth’s radius is 6400 km, find the following.
   (a) angular speed of Earth in radians per day and radians per hour
   (b) linear speed at the North Pole or South Pole
   (c) linear speed at Quito, Ecuador, a city on the equator
   (d) linear speed at Salem, Oregon (halfway from the equator to the North Pole)

95. **Speeds of a Pulley and a Belt** The pulley shown has a radius of 12.96 cm. Suppose it takes 18 sec for 56 cm of belt to go around the pulley.
   (a) Find the angular speed of the belt in radians per second.
   (b) Find the linear speed of the belt in centimeters per second.

96. **Angular Speed of Pulleys** The two pulleys in the figure have radii of 15 cm and 8 cm, respectively. The larger pulley rotates 25 times in 36 sec. Find the angular speed of each pulley in radians per second.
6.3 Graphs of the Sine and Cosine Functions

Periodic Functions

Many things in daily life repeat with a predictable pattern: in warm areas electricity use goes up in summer and down in winter, the price of fresh fruit goes down in summer and up in winter, and attendance at amusement parks increases in spring and declines in autumn. Because the sine and cosine functions repeat their values in a regular pattern, they are **periodic functions**. Figure 19 shows a sine graph that represents a normal heartbeat.

![Figure 19](image)

**Looking Ahead to Calculus**

Periodic functions are used throughout calculus, so you will need to know their characteristics. One use of these functions is to describe the location of a point in the plane using **polar coordinates**, an alternative to rectangular coordinates. (See Chapter 8).

**Periodic Function**

A periodic function is a function $f$ such that

$$f(x) = f(x + np),$$

for every real number $x$ in the domain of $f$, every integer $n$, and some positive real number $p$. The smallest possible positive value of $p$ is the period of the function.

The circumference of the unit circle is $2\pi$, so the smallest value of $p$ for which the sine and cosine functions repeat is $2\pi$. Therefore, the sine and cosine functions are periodic functions with period $2\pi$.
Graph of the Sine Function  In Section 6.1 we saw that for a real number s, the point on the unit circle corresponding to s has coordinates (cos s, sin s). See Figure 20. Trace along the circle to verify the results shown in the table.

<table>
<thead>
<tr>
<th>As s Increases from</th>
<th>sin s</th>
<th>cos s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to $\pi/2$</td>
<td>Increases from 0 to 1</td>
<td>Decreases from 1 to 0</td>
</tr>
<tr>
<td>$\pi/2$ to $\pi$</td>
<td>Decreases from 1 to 0</td>
<td>Decreases from 0 to $-1$</td>
</tr>
<tr>
<td>$\pi$ to $3\pi/2$</td>
<td>Decreases from 0 to $-1$</td>
<td>Increases from $-1$ to 0</td>
</tr>
<tr>
<td>$3\pi/2$ to $2\pi$</td>
<td>Increases from $-1$ to 0</td>
<td>Increases from 0 to 1</td>
</tr>
</tbody>
</table>

To avoid confusion when graphing the sine function, we use $x$ rather than $s$; this corresponds to the letters in the $xy$-coordinate system. Selecting key values of $x$ and finding the corresponding values of sin $x$ leads to the table in Figure 21. To obtain the traditional graph in Figure 21, we plot the points from the table, use symmetry, and join them with a smooth curve. Since $y = \sin x$ is periodic with period $2\pi$ and has domain $(-\infty, \infty)$, the graph continues in the same pattern in both directions. This graph is called a sine wave or sinusoid.

**SINE FUNCTION  $f(x) = \sin x$**

Domain: $(-\infty, \infty)$  Range: $[-1, 1]$  

- The graph is continuous over its entire domain, $(-\infty, \infty)$.
- Its $x$-intercepts are of the form $n\pi$, where $n$ is an integer.
- Its period is $2\pi$.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all $x$ in the domain, $\sin(-x) = -\sin x$. 

![Figure 20](image-url)  

![Figure 21](image-url)
558  CHAPTER 6  The Circular Functions and Their Graphs

Looking Ahead to Calculus
The discussion of the derivative of a function in calculus shows that for the sine function, the slope of the tangent line at any point $x$ is given by $\cos x$. For example, look at the graph of $y = \sin x$ and notice that a tangent line at $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots$ will be horizontal and thus have slope 0. Now look at the graph of $y = \cos x$ and see that for these values, $\cos x = 0$.

Graph of the Cosine Function  We find the graph of $y = \cos x$ in much the same way as the graph of $y = \sin x$. In the table of values shown with Figure 22 for $y = \cos x$, we use the same values for $x$ as we did for the graph of $y = \sin x$. Notice that the graph of $y = \cos x$ in Figure 22 has the same shape as the graph of $y = \sin x$. It is, in fact, the graph of the sine function shifted, or translated, $\frac{\pi}{2}$ units to the left.

COSINE FUNCTION  $f(x) = \cos x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{\pi}{2}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

- The graph is continuous over its entire domain, $(-\infty, \infty)$.
- Its $x$-intercepts are of the form $(2n + 1)\frac{\pi}{2}$, where $n$ is an integer.
- Its period is $2\pi$.
- The graph is symmetric with respect to the $y$-axis, so the function is an even function. For all $x$ in the domain, $\cos(-x) = \cos x$.

Notice that the calculator graphs of $f(x) = \sin x$ in Figure 21 and $f(x) = \cos x$ in Figure 22 are graphed in the window $[-2\pi, 2\pi]$ by $[-4, 4]$, with $Xscl = \frac{\pi}{2}$ and $Yscl = 1$. This is called the trig viewing window. (Your model may use a different “standard” trigonometric viewing window. Consult your owner’s manual.)

Graphing Techniques, Amplitude, and Period  The examples that follow show graphs that are “stretched” or “compressed” either vertically, horizontally, or both when compared with the graphs of $y = \sin x$ or $y = \cos x$. 
6.3 Graphs of the Sine and Cosine Functions

**Example 1.** Graphing \( y = a \sin x \)

Graph \( y = 2 \sin x \), and compare to the graph of \( y = \sin x \).

**Solution.** For a given value of \( x \), the value of \( y \) is twice as large as it would be for \( y = \sin x \), as shown in the table of values. The only change in the graph is the range, which becomes \([-2, 2]\). See Figure 23, which includes a graph of \( y = \sin x \) for comparison.

![Graph of \( y = 2 \sin x \)](image)

The **amplitude** of a periodic function is half the difference between the maximum and minimum values. Thus, for both the basic sine and cosine functions, the amplitude is

\[
\frac{1}{2} \left[ 1 - (-1) \right] = \frac{1}{2} (2) = 1.
\]

Generalizing from Example 1 gives the following.

**Amplitude**

The graph of \( y = a \sin x \) or \( y = a \cos x \), with \( a \neq 0 \), will have the same shape as the graph of \( y = \sin x \) or \( y = \cos x \), respectively, except with range \([-|a|, |a|]\). The amplitude is \(|a|\).

Now try Exercise 7.

---

**Teaching Tip.** Students may think that the period of \( y = \sin 2x \) is \(4\pi\). Explain that a factor of 2 causes values of the argument to increase twice as fast, thereby shortening the length of a period.
These values suggest that a complete cycle is achieved in \( \frac{\pi}{2} \) or \( \frac{2\pi}{2} \) units, which is reasonable since
\[
\sin \left( 4 \cdot \frac{\pi}{2} \right) = \sin 2\pi = 0.
\]

In general, the graph of a function of the form \( y = \sin bx \) or \( y = \cos bx \), for \( b > 0 \), will have a period different from \( 2\pi \) when \( b \neq 1 \). To see why this is so, remember that the values of \( \sin bx \) or \( \cos bx \) will take on all possible values as \( bx \) ranges from 0 to \( 2\pi \). Therefore, to find the period of either of these functions, we must solve the three-part inequality
\[
0 \leq bx \leq 2\pi \quad \text{(Section 1.7)}
\]
\[
0 \leq x \leq \frac{2\pi}{b}.
\]
Divide by the positive number \( b \).

Thus, the period is \( \frac{2\pi}{b} \). By dividing the interval \([0, \frac{2\pi}{b}]\) into four equal parts, we obtain the values for which \( \sin bx \) or \( \cos bx \) is \(-1, 0, \) or \(1\). These values will give minimum points, \( x \)-intercepts, and maximum points on the graph. Once these points are determined, we can sketch the graph by joining the points with a smooth sinusoidal curve. (If a function has \( b < 0 \), then the identities of the next chapter can be used to rewrite the function so that \( b > 0 \).)

**NOTE** One method to divide an interval into four equal parts is as follows.

*Step 1* Find the midpoint of the interval by adding the \( x \)-values of the endpoints and dividing by 2.

*Step 2* Find the midpoints of the two intervals found in Step 1, using the same procedure.

**EXAMPLE 2** Graphing \( y = \sin bx \)

Graph \( y = \sin 2x \), and compare to the graph of \( y = \sin x \).

**Solution** In this function the coefficient of \( x \) is 2, so the period is \( \frac{2\pi}{2} = \pi \). Therefore, the graph will complete one period over the interval \([0, \pi]\).

The endpoints are 0 and \( \pi \), and the three middle points are
\[
\frac{1}{4}(0 + \pi), \quad \frac{1}{2}(0 + \pi), \quad \text{and} \quad \frac{3}{4}(0 + \pi),
\]
which give the following \( x \)-values.
\[
0, \quad \frac{\pi}{4}, \quad \frac{\pi}{2}, \quad \frac{3\pi}{4}, \quad \pi
\]

We plot the points from the table of values given on page 559, and join them with a smooth sinusoidal curve. More of the graph can be sketched by repeating this cycle, as shown in Figure 24. The amplitude is not changed. The graph of \( y = \sin x \) is included for comparison.
6.3 Graphs of the Sine and Cosine Functions

Now try Exercise 15.

Generalizing from Example 2 leads to the following result.

**Period**

For $b > 0$, the graph of $y = \sin bx$ will resemble that of $y = \sin x$, but with period $\frac{2\pi}{b}$. Also, the graph of $y = \cos bx$ will resemble that of $y = \cos x$, but with period $\frac{2\pi}{b}$.

**EXAMPLE 3** Graphing $y = \cos bx$

Graph $y = \cos \frac{2}{3}x$ over one period.

**Solution** The period is $\frac{2\pi}{\frac{2}{3}} = 3\pi$. We divide the interval $[0, 3\pi]$ into four equal parts to get the following $x$-values that yield minimum points, maximum points, and $x$-intercepts.

$$0, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4}, 3\pi$$

We use these values to obtain a table of key points for one period.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$0$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$\frac{9\pi}{4}$</th>
<th>$3\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{3}x$</td>
<td>0</td>
<td>$\frac{\pi}{4}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{4}$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$\frac{9\pi}{4}$</td>
<td>3$\pi$</td>
</tr>
<tr>
<td>$\cos \frac{1}{3}x$</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The amplitude is 1 because the maximum value is 1, the minimum value is $-1$, and half of $1 - (-1)$ is $\frac{1}{2}(2) = 1$. We plot these points and join them with a smooth curve. The graph is shown in Figure 25.

Now try Exercise 13.

**NOTE** Look at the middle row of the table in Example 3. The method of dividing the interval $[0, \frac{2\pi}{b}]$ into four equal parts will always give the values $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ and } 2\pi$ for this row, resulting in values of $-1, 0, \text{ or } 1$ for the circular function. These lead to key points on the graph, which can then be easily sketched.
The method used in Examples 1–3 is summarized as follows.

**Guidelines for Sketching Graphs of Sine and Cosine Functions**

To graph \( y = a \sin bx \) or \( y = a \cos bx \), with \( b > 0 \), follow these steps.

**Step 1**  Find the period, \( \frac{2\pi}{b} \). Start at 0 on the \( x \)-axis, and lay off a distance of \( \frac{2\pi}{b} \).

**Step 2**  Divide the interval into four equal parts. (See the Note preceding Example 2.)

**Step 3**  Evaluate the function for each of the five \( x \)-values resulting from Step 2. The points will be maximum points, minimum points, and \( x \)-intercepts.

**Step 4**  Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude \( |a| \).

**Step 5**  Draw the graph over additional periods, to the right and to the left, as needed.

The function in Example 4 has both amplitude and period affected by the values of \( a \) and \( b \).

**EXAMPLE 4  Graphing \( y = a \sin bx \)**

Graph \( y = -2 \sin 3x \) over one period using the preceding guidelines.

**Solution**

**Step 1**  For this function, \( b = 3 \), so the period is \( \frac{2\pi}{3} \). The function will be graphed over the interval \( \left[ 0, \frac{2\pi}{3} \right] \).

**Step 2**  Divide the interval \( \left[ 0, \frac{2\pi}{3} \right] \) into four equal parts to get the \( x \)-values 0, \( \frac{\pi}{3} \), \( \frac{\pi}{2} \), \( \frac{2\pi}{3} \), and \( \pi \).

**Step 3**  Make a table of values determined by the \( x \)-values from Step 2.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \frac{2\pi}{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3x )</td>
<td>0</td>
<td>( \frac{\pi}{2} )</td>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
<td>2\pi</td>
</tr>
<tr>
<td>( \sin 3x )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( -2 \sin 3x )</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

**Step 4**  Plot the points \( (0, 0), \left( \frac{\pi}{6}, -2 \right), \left( \frac{\pi}{3}, 0 \right), \left( \frac{\pi}{2}, 2 \right), \) and \( \left( \frac{2\pi}{3}, 0 \right) \), and join them with a sinusoidal curve with amplitude 2. See Figure 26.

**Step 5**  The graph can be extended by repeating the cycle.

Notice that when \( a \) is negative, the graph of \( y = a \sin bx \) is the reflection across the \( x \)-axis of the graph of \( y = |a| \sin bx \).

Now try Exercise 19.
6.3 Graphs of the Sine and Cosine Functions

Translations

In general, the graph of the function defined by \( y = f(x - d) \) is translated horizontally when compared to the graph of \( y = f(x) \). The translation is \( d \) units to the right if \( d > 0 \) and \( |d| \) units to the left if \( d < 0 \). See Figure 27. With trigonometric functions, a horizontal translation is called a phase shift. In the function \( y = f(x - d) \), the expression \( x - d \) is called the argument.

In Example 5, we give two methods that can be used to sketch the graph of a circular function involving a phase shift.

**EXAMPLE 5** Graphing \( y = \sin(x - d) \)

Graph \( y = \sin\left(x - \frac{\pi}{3}\right) \).

**Solution**  
*Method 1* For the argument \( x - \frac{\pi}{3} \) to result in all possible values throughout one period, it must take on all values between 0 and \( 2\pi \), inclusive. Therefore, to find an interval of one period, we solve the three-part inequality

\[
0 \leq x - \frac{\pi}{3} \leq 2\pi
\]

\[
\frac{\pi}{3} \leq x \leq \frac{7\pi}{3}.
\]

Add \( \frac{\pi}{3} \) to each part.

Divide the interval \( \left[\frac{\pi}{3}, \frac{7\pi}{3}\right] \) into four equal parts to get the following \( x \)-values.

\[
\frac{\pi}{3}, \quad \frac{5\pi}{6}, \quad \frac{4\pi}{3}, \quad \frac{11\pi}{6}, \quad \frac{7\pi}{3}
\]

A table of values using these \( x \)-values follows.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{5\pi}{6} )</th>
<th>( \frac{4\pi}{3} )</th>
<th>( \frac{11\pi}{6} )</th>
<th>( \frac{7\pi}{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x - \frac{\pi}{3} )</td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
<td>( 2\pi )</td>
</tr>
<tr>
<td>( \sin(x - \frac{\pi}{3}) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

We join the corresponding points to get the graph shown in Figure 28. The period is \( 2\pi \), and the amplitude is 1.

*Method 2* We can also graph \( y = \sin(x - \frac{\pi}{3}) \) using a horizontal translation. The argument \( x - \frac{\pi}{3} \) indicates that the graph will be translated \( \frac{\pi}{3} \) units to the right (the phase shift) as compared to the graph of \( y = \sin x \). In Figure 28 we show the graph of \( y = \sin x \) as a dashed curve, and the graph of \( y = \sin(x - \frac{\pi}{3}) \) as a solid curve. Therefore, to graph a function using this method, first graph the basic circular function, and then graph the desired function by using the appropriate translation.

The graph can be extended through additional periods by repeating this portion of the graph over and over, as necessary.

Now try Exercise 37.
The graph of a function of the form \( y = c + f(x) \) is translated vertically as compared with the graph of \( y = f(x) \). See Figure 29. The translation is \( c \) units up if \( c > 0 \) and \( |c| \) units down if \( c < 0 \).

Vertical translations of \( y = f(x) \)

**EXAMPLE 6** Graphing \( y = c + a \cos bx \)

Graph \( y = 3 - 2 \cos 3x \).

**Solution** The values of \( y \) will be 3 greater than the corresponding values of \( y \) in \( y = -2 \cos 3x \). This means that the graph of \( y = 3 - 2 \cos 3x \) is the same as the graph of \( y = -2 \cos 3x \), vertically translated 3 units up. Since the period of \( y = -2 \cos 3x \) is \( \frac{2\pi}{3} \), the key points have \( x \)-values

\[
0, \quad \frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{2\pi}{3}
\]

Use these \( x \)-values to make a table of points.

| \( x \) | 0 | \( \frac{\pi}{6} \) | \( \frac{\pi}{3} \) | \( \frac{\pi}{2} \) | \( \frac{2\pi}{3} \) |
|---|---|---|---|---|
| \( \cos 3x \) | 1 | 0 | -1 | 0 | 1 |
| \( 2 \cos 3x \) | 2 | 0 | -2 | 0 | 2 |
| \( 3 - 2 \cos 3x \) | 1 | 3 | 5 | 3 | 1 |

The key points are shown on the graph in Figure 30, along with more of the graph, sketched using the fact that the function is periodic.

Now try Exercise 45.
Combinations of Translations A function of the form
\[ y = c + a \sin bx \quad \text{or} \quad y = c + a \cos bx, \quad b > 0, \]
can be graphed according to the following guidelines.

Further Guidelines for Sketching Graphs of Sine and Cosine Functions

<table>
<thead>
<tr>
<th>Method 1</th>
<th>Follow these steps.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>Find an interval whose length is one period ( \frac{2\pi}{b} ) by solving the three-part inequality ( 0 \leq bx - d \leq 2\pi ).</td>
</tr>
<tr>
<td>Step 2</td>
<td>Divide the interval into four equal parts.</td>
</tr>
<tr>
<td>Step 3</td>
<td>Evaluate the function for each of the five x-values resulting from Step 2. The points will be maximum points, minimum points, and points that intersect the line ( y = c ) (“middle” points of the wave).</td>
</tr>
<tr>
<td>Step 4</td>
<td>Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude (</td>
</tr>
<tr>
<td>Step 5</td>
<td>Draw the graph over additional periods, to the right and to the left, as needed.</td>
</tr>
</tbody>
</table>

Method 2 First graph the basic circular function. The amplitude of the function is \( |a| \), and the period is \( \frac{2\pi}{b} \). Then use translations to graph the desired function. The vertical translation is \( c \) units up if \( c > 0 \) and \( |c| \) units down if \( c < 0 \). The horizontal translation (phase shift) is \( d \) units to the right if \( d > 0 \) and \( |d| \) units to the left if \( d < 0 \).

EXAMPLE 7 Graphing \( y = c + a \sin bx(d) \)

Graph \( y = -1 + 2 \sin(4x + \pi) \).

Solution We use Method 1. First write the expression in the form \( c + a \sin bx(d) \) by rewriting \( 4x + \pi \) as \( 4(x + \frac{\pi}{4}) \):

\[ y = -1 + 2 \sin \left(4 \left(x + \frac{\pi}{4}\right)\right). \]

Rewrite \( 4x + \pi \) as \( 4(x + \frac{\pi}{4}) \).

Step 1 Find an interval whose length is one period.

\[
0 \leq 4\left(x + \frac{\pi}{4}\right) \leq 2\pi
\]

\[
0 \leq x + \frac{\pi}{4} \leq \frac{\pi}{2}
\]

Divide by 4.

\[
-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}
\]

Subtract \( \frac{\pi}{4} \).

Step 2 Divide the interval \( \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \) into four equal parts to get the x-values

\[
-\frac{\pi}{4}, \quad -\frac{\pi}{8}, \quad 0, \quad \frac{\pi}{8}, \quad \frac{\pi}{4}.
\]
**Step 3** Make a table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\frac{\pi}{4}$</th>
<th>$-\frac{\pi}{8}$</th>
<th>$0$</th>
<th>$\frac{\pi}{8}$</th>
<th>$\frac{\pi}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + \frac{\pi}{4}$</td>
<td>0</td>
<td>$\frac{\pi}{8}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{3\pi}{8}$</td>
<td>$\frac{\pi}{2}$</td>
</tr>
<tr>
<td>$4(x + \frac{\pi}{4})$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\sin 4(x + \frac{\pi}{4})$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$2 \sin 4(x + \frac{\pi}{4})$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$-2$</td>
<td>0</td>
</tr>
<tr>
<td>$-1 + 2 \sin(4x + \pi)$</td>
<td>$-1$</td>
<td>1</td>
<td>$-1$</td>
<td>$-3$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

**Steps 4 and 5** Plot the points found in the table and join them with a sinusoidal curve. Figure 31 shows the graph, extended to the right and left to include two full periods.

Now try Exercise 49.

### Determining a Trigonometric Model Using Curve Fitting

A sinusoidal function is often a good approximation of a set of real data points.

**EXAMPLE 8** Modeling Temperature with a Sine Function

The maximum average monthly temperature in New Orleans is 82°F and the minimum is 54°F. The table shows the average monthly temperatures. The scatter diagram for a 2-year interval in Figure 32 strongly suggests that the temperatures can be modeled with a sine curve.

<table>
<thead>
<tr>
<th>Month</th>
<th>°F</th>
<th>Month</th>
<th>°F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>54</td>
<td>July</td>
<td>82</td>
</tr>
<tr>
<td>Feb</td>
<td>55</td>
<td>Aug</td>
<td>81</td>
</tr>
<tr>
<td>Mar</td>
<td>61</td>
<td>Sept</td>
<td>77</td>
</tr>
<tr>
<td>Apr</td>
<td>69</td>
<td>Oct</td>
<td>71</td>
</tr>
<tr>
<td>May</td>
<td>73</td>
<td>Nov</td>
<td>59</td>
</tr>
<tr>
<td>June</td>
<td>79</td>
<td>Dec</td>
<td>55</td>
</tr>
</tbody>
</table>


(a) Using only the maximum and minimum temperatures, determine a function of the form $f(x) = a \sin [b(x - d)] + c$, where $a$, $b$, $c$, and $d$ are constants, that models the average monthly temperature in New Orleans. Let $x$ represent the month, with January corresponding to $x = 1$. 

![Figure 31](image-url)  

![Figure 32](image-url)
(b) On the same coordinate axes, graph \( f \) for a two-year period together with the actual data values found in the table.

(c) Use the sine regression feature of a graphing calculator to determine a second model for these data.

**Solution**

(a) We use the maximum and minimum average monthly temperatures to find the amplitude \( a \).

\[
a = \frac{82 - 54}{2} = 14
\]

The average of the maximum and minimum temperatures is a good choice for \( c \). The average is

\[
\frac{82 + 54}{2} = 68.
\]

Since the coldest month is January, when \( x = 1 \), and the hottest month is July, when \( x = 7 \), we should choose \( d \) to be about 4. We experiment with values just greater than 4 to find \( d \). Trial and error using a calculator leads to \( d = 4.2 \). Since temperatures repeat every 12 months, \( b \) is \( \frac{2\pi}{12} = \frac{\pi}{6} \). Thus,

\[
f(x) = a \sin[b(x - d)] + c = 14 \sin\left[\frac{\pi}{6}(x - 4.2)\right] + 68.
\]

(b) Figure 33 shows the data points and the graph of \( y = 14 \sin \frac{\pi}{6}x + 68 \) for comparison. The horizontal translation of the model is fairly obvious here.

(c) We used the given data for a two-year period to produce the model described in Figure 34(a). Figure 34(b) shows its graph along with the data points.

Now try Exercise 73.
6.3 Exercises

Concept Check  In Exercises 1–4, match each function with its graph.

1. \( y = -\sin x \)  
   A. 

2. \( y = -\cos x \)  
   B. 

3. \( y = \sin 2x \)  
   C. 

4. \( y = 2\cos x \)  
   D. 

Graph each function over the interval \([-2\pi, 2\pi]\). Give the amplitude. See Example 1.

5. \( y = 2\cos x \)  
6. \( y = 3\sin x \)  
7. \( y = \frac{2}{3}\sin x \)  
8. \( y = \frac{3}{4}\cos x \)  
9. \( y = -\cos x \)  
10. \( y = -\sin x \)  

Graph each function over a two-period interval. Give the period and amplitude. See Examples 2–4.

11. \( y = -2\sin x \)  
12. \( y = -3\cos x \)  
13. \( y = \sin \frac{1}{2}x \)  
14. \( y = \sin \frac{2}{3}x \)  
15. \( y = \cos 2x \)  
16. \( y = \cos \frac{3}{4}x \)  
17. \( y = 2\sin \frac{1}{4}x \)  
18. \( y = 3\sin 2x \)  
19. \( y = -2\cos 3x \)  
20. \( y = -5\cos 2x \)  

Concept Check  In Exercises 21 and 22, give the equation of a sine function having the given graph.

21. 

22. 

WINDOW
\( Xmin=-20 \)
\( Xmax=20 \)
\( Xscl=5 \)
\( Ymin=-5 \)
\( Ymax=5 \)
\( Yscl=1 \)
\( Xres=1 \)
6.3 Graphs of the Sine and Cosine Functions

**Concept Check**  
Match each function in Column I with the appropriate description in Column II.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 3 \sin(2x - 4)$</td>
<td>A. amplitude = $2$, period = $\frac{\pi}{2}$, phase shift = $\frac{3}{4}$</td>
</tr>
<tr>
<td>$y = 2 \sin(3x - 4)$</td>
<td>B. amplitude = $3$, period = $\pi$, phase shift = $2$</td>
</tr>
<tr>
<td>$y = 4 \sin(3x - 2)$</td>
<td>C. amplitude = $4$, period = $\frac{2\pi}{3}$, phase shift = $\frac{2}{3}$</td>
</tr>
<tr>
<td>$y = 2 \sin(4x - 3)$</td>
<td>D. amplitude = $2$, period = $\frac{2\pi}{3}$, phase shift = $\frac{4}{3}$</td>
</tr>
</tbody>
</table>

**Concept Check**  
Match each function with its graph.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \sin \left( x - \frac{\pi}{4} \right)$</td>
<td>A.</td>
</tr>
<tr>
<td>$y = \sin \left( x + \frac{\pi}{4} \right)$</td>
<td>B.</td>
</tr>
<tr>
<td>$y = 1 + \sin x$</td>
<td>C.</td>
</tr>
<tr>
<td>$y = -1 + \sin x$</td>
<td>D.</td>
</tr>
</tbody>
</table>

Find the amplitude, the period, any vertical translation, and any phase shift of the graph of each function. See Examples 5–7.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 2 \sin(x - \pi)$</td>
<td>32. $y = \frac{2}{3} \sin \left( x + \frac{\pi}{2} \right)$</td>
</tr>
<tr>
<td>$y = 4 \cos \left( \frac{x}{2} + \frac{\pi}{2} \right)$</td>
<td>33. $y = \frac{1}{2} \sin \left( \frac{x}{2} + \pi \right)$</td>
</tr>
<tr>
<td>$y = 2 - \sin \left( 3x - \frac{\pi}{5} \right)$</td>
<td>34. $y = -1 + \frac{1}{2} \cos(2x - 3\pi)$</td>
</tr>
<tr>
<td>$y = 2 \cos \left( x - \frac{\pi}{3} \right)$</td>
<td>35. $y = 3 \sin \left( x - \frac{3\pi}{2} \right)$</td>
</tr>
</tbody>
</table>

Graph each function over a two-period interval. See Example 5.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \sin \left( x - \frac{\pi}{4} \right)$</td>
<td>36. $y = \cos \left( x - \frac{\pi}{3} \right)$</td>
</tr>
<tr>
<td>$y = 2 \cos \left( x - \frac{\pi}{3} \right)$</td>
<td>37. $y = 3 \sin \left( x - \frac{3\pi}{2} \right)$</td>
</tr>
</tbody>
</table>

Graph each function over a one-period interval.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = -4 \sin(2x - \pi)$</td>
<td>41. $y = 3 \cos(4x + \pi)$</td>
</tr>
<tr>
<td>$y = \frac{1}{2} \cos \left( \frac{1}{2} x - \frac{\pi}{4} \right)$</td>
<td>42. $y = -\frac{1}{4} \sin \left( \frac{3}{4} x + \frac{\pi}{8} \right)$</td>
</tr>
</tbody>
</table>

Graph each function over a two-period interval. See Example 6.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = -1 - 2 \cos 5x$</td>
<td>45. $y = -1 - 2 \cos 5x$</td>
</tr>
<tr>
<td>$y = 1 - \frac{2}{3} \sin \frac{3}{4} x$</td>
<td>46. $y = 1 - \frac{2}{3} \sin \frac{3}{4} x$</td>
</tr>
</tbody>
</table>
Graph each function over a one-period interval. See Example 7.

\begin{align*}
47. & \quad y = 1 - 2 \cos \frac{1}{2}x \\
48. & \quad y = -3 + 3 \sin \frac{1}{2}x \\
49. & \quad y = -3 + 2 \sin \left( x + \frac{\pi}{2} \right) \\
50. & \quad y = 4 - 3 \cos(x - \pi) \\
51. & \quad y = \frac{1}{2} + \sin \left( x + \frac{\pi}{2} \right) \\
52. & \quad y = -\frac{5}{2} + \cos \left( x - \frac{\pi}{6} \right)
\end{align*}

Concept Check

In Exercises 53 and 54, find the equation of a sine function having the given graph.

53. (Note: $\text{Xscl} = \frac{\pi}{4}$.)

\begin{align*}
\text{WINDOW} \\
\text{Xmin} &= -5 \\
\text{Xmax} &= 5 \\
\text{Xscl} &= 78539816... \\
\text{Ymin} &= -5 \\
\text{Ymax} &= 5 \\
\text{Yscl} &= 1 \\
\text{Xres} &= 1
\end{align*}

54. (Note: $\text{Yscl} = \pi$.)

\begin{align*}
\text{WINDOW} \\
\text{Xmin} &= -3 \\
\text{Xmax} &= 3 \\
\text{Xscl} &= 3.1415926... \\
\text{Ymin} &= -3 \\
\text{Ymax} &= 3 \\
\text{Yscl} &= 1 \\
\text{Xres} &= 1
\end{align*}

(Solving) Solve each problem.

55. **Average Annual Temperature**

Scientists believe that the average annual temperature in a given location is periodic. The average temperature at a given place during a given season fluctuates as time goes on, from colder to warmer, and back to colder. The graph shows an idealized description of the temperature (in °F) for the last few thousand years of a location at the same latitude as Anchorage, Alaska.

\[ \text{Average Annual Temperature (Idealized)} \]

\[ \text{Years ago} \]

(a) Find the highest and lowest temperatures recorded.

(b) Use these two numbers to find the amplitude.

(c) Find the period of the function.

(d) What is the trend of the temperature now?
56. Blood Pressure Variation  The graph gives the variation in blood pressure for a typical person. Systolic and diastolic pressures are the upper and lower limits of the periodic changes in pressure that produce the pulse. The length of time between peaks is called the period of the pulse.

(a) Find the amplitude of the graph.
(b) Find the pulse rate (the number of pulse beats in 1 min) for this person.

57. Activity of a Nocturnal Animal  Many of the activities of living organisms are periodic. For example, the graph below shows the time that a certain nocturnal animal begins its evening activity.

(a) Find the amplitude of this graph.
(b) Find the period.

58. Position of a Moving Arm  The figure shows schematic diagrams of a rhythmically moving arm. The upper arm $RO$ rotates back and forth about the point $R$; the position of the arm is measured by the angle $y$ between the actual position and the downward vertical position. (Source: De Sapio, Rodolfo, Calculus for the Life Sciences. Copyright © 1978 by W. H. Freeman and Company. Reprinted by permission.)

(a) Find an equation of the form $y = a \sin kt$ for the graph shown.
(b) How long does it take for a complete movement of the arm?
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59. 24 hr
60. approximately $\frac{2.6 - 0.2}{2} = 1.2$
61. approximately 6:00 P.M.; approximately 0.2 ft
62. approximately 7:19 P.M.; approximately 0 ft
63. approximately 2:00 A.M.; approximately 2.6 ft
64. approximately 3:18 A.M.; approximately 2.4 ft
65. 1; $\frac{4\pi}{3}$
66. 1; $\frac{2\pi}{3}$
67. (a) $\frac{1}{60}$; (b) 60
   (c) 5: 1.545; -0.045; -0.045; 1.545
   (d) $E = 5 \cos 120\pi t$

Tides for Kahului Harbor  The chart shows the tides for Kahului Harbor (on the island of Maui, Hawaii). To identify high and low tides and times for other Maui areas, the following adjustments must be made.

Hana: High, +0.40 min, +0.1 ft; Makena: High, +1:21, -0.5 ft;
Low, +18 min, -0.2 ft Low, +1:09, -0.2 ft
Maalaea: High, +1:52, -0.1 ft; Lahaina: High, +1:18, -0.2 ft;
Low, +1:19, -0.2 ft Low, +1:01, -0.1 ft

Use the graph to work Exercises 59–64.

59. The graph is an example of a periodic function. What is the period (in hours)?
60. What is the amplitude?
61. At what time on January 20 was low tide at Kahului? What was the height?
62. Repeat Exercise 61 for Maalaea.
63. At what time on January 22 was high tide at Kahului? What was the height?
64. Repeat Exercise 63 for Lahaina.

Musical Sound Waves  Pure sounds produce single sine waves on an oscilloscope. Find the amplitude and period of each sine wave graph in Exercises 65 and 66. On the vertical scale, each square represents 0.5; on the horizontal scale, each square represents $\frac{30\text{ sec}}{\pi}$.

65. 66.

(Modeling)  Solve each problem.

67. Voltage of an Electrical Circuit  The voltage $E$ in an electrical circuit is modeled by

$$E = 5 \cos 120\pi t,$$

where $t$ is time measured in seconds.

(a) Find the amplitude and the period.
(b) How many cycles are completed in 1 sec? (The number of cycles (periods) completed in 1 sec is the frequency of the function.)
(c) Find $E$ when $t = 0, 0.03, 0.06, 0.09, 0.12$.
(d) Graph $E$ for $0 \leq t \leq \frac{1}{30}$.
68. **Voltage of an Electrical Circuit** For another electrical circuit, the voltage $E$ is modeled by

$$E = 3.8 \cos 40\pi t,$$

where $t$ is time measured in seconds.

(a) Find the amplitude and the period.
(b) Find the frequency. See Exercise 67(b).
(c) Find $E$ when $t = .02, .04, .08, .12, .14$.
(d) Graph one period of $E$.

69. **Atmospheric Carbon Dioxide** At Mauna Loa, Hawaii, atmospheric carbon dioxide levels in parts per million (ppm) have been measured regularly since 1958. The function defined by

$$L(x) = .022x^2 + .55x + 316 + 3.5 \sin(2\pi x)$$

can be used to model these levels, where $x$ is in years and $x = 0$ corresponds to 1960. (Source: Nilssson, A., *Greenhouse Earth*, John Wiley & Sons, 1992.)

(a) Graph $L$ in the window $[15, 35]$ by $[325, 365]$.
(b) When do the seasonal maximum and minimum carbon dioxide levels occur?
(c) $L$ is the sum of a quadratic function and a sine function. What is the significance of each of these functions? Discuss what physical phenomena may be responsible for each function.

70. **Atmospheric Carbon Dioxide** Refer to Exercise 69. The carbon dioxide content in the atmosphere at Barrow, Alaska, in parts per million (ppm) can be modeled using the function defined by

$$C(x) = .04x^2 + .6x + 330 + 7.5 \sin(2\pi x),$$


(a) Graph $C$ in the window $[5, 25]$ by $[320, 380]$.
(b) Discuss possible reasons why the amplitude of the oscillations in the graph of $C$ is larger than the amplitude of the oscillations in the graph of $L$ in Exercise 69, which models Hawaii.
(c) Define a new function $C$ that is valid if $x$ represents the actual year, where $1970 \leq x \leq 1995$.

71. **Temperature in Fairbanks** The temperature in Fairbanks is modeled by

$$T(x) = 37 \sin \left( \frac{2\pi}{365}(x - 101) \right) + 25,$$

where $T(x)$ is the temperature in degrees Fahrenheit on day $x$, with $x = 1$ corresponding to January 1 and $x = 365$ corresponding to December 31. Use a calculator to estimate the temperature on the following days. (Source: Lando, B. and C. Lando, "Is the Graph of Temperature Variation a Sine Curve?", *The Mathematics Teacher*, 70, September 1977.)

(a) March 1 (day 60)  
(b) April 1 (day 91)  
(c) Day 150  
(d) June 15  
(e) September 1  
(f) October 31

72. **Fluctuation in the Solar Constant** The solar constant $S$ is the amount of energy per unit area that reaches Earth's atmosphere from the sun. It is equal to 1367 watts
72. (a) 1.998 watts per m² (b) -46.461 watts per m² (c) 46.478 watts per m² (d) Answers may vary. A possible answer is \( N = 82.5 \). (Since \( N \) represents a day number, which should be a natural number, we might interpret day 82.5 as noon on the 82nd day.) Answer graphs for Exercises 73(a), (b), and (e) are included on page A-38 of the answer section at the back of the text.

73. (a) yes (b) It represents the average yearly temperature. (c) 14; 12; 4.2 (d) \( f(x) = 14 \sin \left( \frac{\pi}{6} (x - 4.2) \right) + 50 \) (e) The function gives an excellent model for the given data.

\[ f(x) = \sin \left( \frac{\pi}{6} (x - 4.2) \right) + 70.5 \]

The function gives an excellent model for the data.

74. (a) 70.4\(^\circ\F\) (b) See the graph in part (d). (c) \( f(x) = 19.5 \cos \left( \frac{\pi}{6} (x - 7.2) \right) + 70.5 \) (d) The function gives an excellent model for the data. (e) \[ f(x) = 19.5 \cos \left( \frac{\pi}{6} (x - 7.2) \right) + 70.5 \]

73. **Average Monthly Temperature** The average monthly temperature (in °F) in Vancouver, Canada, is shown in the table.

(a) Plot the average monthly temperature over a two-year period letting \( x = 1 \) correspond to the month of January during the first year. Do the data seem to indicate a translated sine graph?

(b) The highest average monthly temperature is 64°F in July, and the lowest average monthly temperature is 36°F in January. Their average is 50°F. Graph the data together with the line \( y = 50 \). What does this line represent with regard to temperature in Vancouver?

(c) Approximate the amplitude, period, and phase shift of the translated sine wave.

(d) Determine a function of the form \( f(x) = a \sin b(x - d) + c \), where \( a, b, c, \) and \( d \) are constants, that models the data.

(e) Graph \( f \) together with the data on the same coordinate axes. How well does \( f \) model the given data?

(f) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data.

74. **Average Monthly Temperature** The average monthly temperature (in °F) in Phoenix, Arizona, is shown in the table.

(a) Predict the average yearly temperature and compare it to the actual value of 70°F.

(b) Plot the average monthly temperature over a two-year period by letting \( x = 1 \) correspond to January of the first year.

(c) Determine a function of the form \( f(x) = a \cos b(x - d) + c \), where \( a, b, c, \) and \( d \) are constants, that models the data.

(d) Graph \( f \) together with the data on the same coordinate axes. How well does \( f \) model the data?

(e) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data.
6.4 Graphs of the Other Circular Functions

Graphs of the Cosecant and Secant Functions

Since cosecant values are reciprocals of the corresponding sine values, the period of the function \( y = \csc x \) is \( 2\pi \), the same as for \( y = \sin x \). When \( \sin x = 1 \), the value of \( \csc x \) is also 1, and when \( 0 < \sin x < 1 \), then \( \csc x > 1 \). Also, if \( -1 < \sin x < 0 \), then \( \csc x < -1 \). (Verify these statements.) As \( |x| \) approaches 0, \( |\sin x| \) approaches 0, and \( |\csc x| \) gets larger and larger. The graph of \( y = \csc x \) approaches the vertical line \( x = 0 \) but never touches it, so the line \( x = 0 \) is a *vertical asymptote*. In fact, the lines \( x = n\pi \), where \( n \) is any integer, are all vertical asymptotes.

Using this information and plotting a few points shows that the graph takes the shape of the solid curve shown in Figure 35. To show how the two graphs are related, the graph of \( y = \sin x \) is shown as a dashed curve.

![Figure 35](image)

A similar analysis for the secant leads to the solid curve shown in Figure 36. The dashed curve, \( y = \cos x \), is shown so that the relationship between these two reciprocal functions can be seen.

Typically, calculators do not have keys for the cosecant and secant functions. To graph \( y = \csc x \) with a graphing calculator, use the fact that

\[
\csc x = \frac{1}{\sin x}.
\]

The graphs of \( Y_1 = \sin X \) and \( Y_2 = \csc X \) are shown in Figure 37. The calculator is in split screen and connected modes. Similarly, the secant function is graphed by using the identity

\[
\sec x = \frac{1}{\cos x},
\]

as shown in Figure 38.

Using dot mode for graphing will eliminate the vertical lines that appear in Figures 37 and 38. While they suggest asymptotes and are sometimes called *pseudo-asymptotes*, they are not actually parts of the graphs. See Figure 39 on the next page, for example.
COSECANT FUNCTION $f(x) = \csc x$

Domain: $\{x | x \neq n\pi, \text{ where } n \text{ is an integer}\}$  
Range: $(-\infty, -1] \cup [1, \infty)$

- The graph is discontinuous at values of $x$ of the form $x = n\pi$ and has vertical asymptotes at these values.
- There are no $x$-intercepts.
- Its period is $2\pi$.
- Its graph has no amplitude, since there are no maximum or minimum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all $x$ in the domain, $\csc(-x) = -\csc x$.

SECANT FUNCTION $f(x) = \sec x$

Domain: $\{x | x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is an integer}\}$  
Range: $(-\infty, -1] \cup [1, \infty)$

(continued)
The graph is discontinuous at values of \( x \) of the form \( x = (2n + 1)\frac{\pi}{2} \) and has vertical asymptotes at these values.

- There are no \( x \)-intercepts.
- Its period is \( 2\pi \).
- Its graph has no amplitude, since there are no maximum or minimum values.
- The graph is symmetric with respect to the \( y \)-axis, so the function is an even function. For all \( x \) in the domain, \( \sec(-x) = \sec x \).

In the previous section, we gave guidelines for sketching graphs of sine and cosine functions. We now present similar guidelines for graphing cosecant and secant functions.

**Guidelines for Sketching Graphs of Cosecant and Secant Functions**

To graph \( y = a \csc bx \) or \( y = a \sec bx \), with \( b > 0 \), follow these steps.

*Step 1* Graph the corresponding reciprocal function as a guide, using a dashed curve.

<table>
<thead>
<tr>
<th>To Graph</th>
<th>Use as a Guide</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = a \csc bx )</td>
<td>( y = a \sin bx )</td>
</tr>
<tr>
<td>( y = a \sec bx )</td>
<td>( y = a \cos bx )</td>
</tr>
</tbody>
</table>

*Step 2* Sketch the vertical asymptotes. They will have equations of the form \( x = k \), where \( k \) is an \( x \)-intercept of the graph of the guide function.

*Step 3* Sketch the graph of the desired function by drawing the typical U-shaped branches between the adjacent asymptotes. The branches will be above the graph of the guide function when the guide function values are positive and below the graph of the guide function when the guide function values are negative. The graph will resemble those in Figures 39 and 40 in the function boxes on the previous page.

Like graphs of the sine and cosine functions, graphs of the secant and cosecant functions may be translated vertically and horizontally. The period of both basic functions is \( 2\pi \).
EXAMPLE 1 Graphing $y = a \sec bx$

Graph $y = 2 \sec \frac{1}{2}x$.

Solution

Step 1 This function involves the secant, so the corresponding reciprocal function will involve the cosine. The guide function to graph is

$$y = 2 \cos \frac{1}{2}x.$$ 

Using the guidelines of Section 6.3, we find that this guide function has amplitude 2 and one period of the graph lies along the interval that satisfies the inequality

$$0 \leq \frac{1}{2}x \leq 2\pi, \quad \text{or} \quad [0, 4\pi]. \quad (\text{Section 1.7})$$

Dividing this interval into four equal parts gives the key points

$$(0, 2), \quad (\pi, 0), \quad (2\pi, -2), \quad (3\pi, 0), \quad (4\pi, 2),$$

which are joined with a smooth dashed curve to indicate that this graph is only a guide. An additional period is graphed as seen in Figure 41(a).

Step 2 Sketch the vertical asymptotes. These occur at $x$-values for which the guide function equals 0, such as

$$x = -3\pi, \quad x = -\pi, \quad x = \pi, \quad x = 3\pi.$$ 

See Figure 41(a).

Step 3 Sketch the graph of $y = 2 \sec \frac{1}{2}x$ by drawing the typical U-shaped branches, approaching the asymptotes. See Figure 41(b).

Now try Exercise 7.
EXAMPLE 2 Graphing \( y = a \csc(x - d) \)

Graph \( y = \frac{3}{2} \csc \left( x - \frac{\pi}{2} \right) \).

Solution

Step 1 Use the guidelines of Section 6.3 to graph the corresponding reciprocal function

\[ y = \frac{3}{2} \sin \left( x - \frac{\pi}{2} \right), \]

shown as a red dashed curve in Figure 42.

Step 2 Sketch the vertical asymptotes through the \( x \)-intercepts of the graph of \( y = \frac{3}{2} \sin(x - \frac{\pi}{2}) \). These have the form \( x = (2n + 1) \frac{\pi}{2} \), where \( n \) is an integer. See the black dashed lines in Figure 42.

Step 3 Sketch the graph of \( y = \frac{3}{2} \csc(x - \frac{\pi}{2}) \) by drawing the typical U-shaped branches between adjacent asymptotes. See the solid blue graph in Figure 42.

Figure 42

Now try Exercise 9.

Graphs of the Tangent and Cotangent Functions Unlike the four functions whose graphs we studied previously, the tangent function has period \( \pi \). Because \( \tan x = \frac{\sin x}{\cos x} \), tangent values are 0 when sine values are 0, and undefined when cosine values are 0. As \( x \)-values go from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\), tangent values go from \(-\infty\) to \(\infty\) and increase throughout the interval. Those same values are repeated as \( x \) goes from \(\frac{\pi}{2}\) to \(\frac{3\pi}{2} \), \(\frac{3\pi}{2}\) to \(\frac{5\pi}{2} \), and so on. The graph of \( y = \tan x \) from \(-\pi\) to \(\frac{3\pi}{2}\) is shown in Figure 43.

Figure 43
The cotangent function also has period \( \pi \). Cotangent values are 0 when cosine values are 0, and undefined when sine values are 0. (Verify this also.) As \( x \)-values go from 0 to \( \pi \), cotangent values go from \( \infty \) to \( -\infty \) and decrease throughout the interval. Those same values are repeated as \( x \) goes from \( \pi \) to \( 2\pi \), \( 2\pi \) to \( 3\pi \), and so on. The graph of \( y = \cot x \) from \( -\pi \) to \( \pi \) is shown in Figure 44.

**TANGENT FUNCTION  \( f(x) = \tan x \)**

Domain: \( \{ x \mid x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is an integer} \} \)  \hspace{1cm} Range: \( (-\infty, \infty) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\frac{\pi}{4} )</td>
<td>undefined</td>
</tr>
<tr>
<td>( -\frac{\pi}{2} )</td>
<td>( -1 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>undefined</td>
</tr>
</tbody>
</table>

\( f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2} \)

- The graph is discontinuous at values of \( x \) of the form \( x = (2n + 1)\frac{\pi}{2} \) and has vertical asymptotes at these values.
- Its \( x \)-intercepts are of the form \( x = n\pi \).
- Its period is \( \pi \).
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all \( x \) in the domain, \( \tan(-x) = -\tan x \).

**COTANGENT FUNCTION  \( f(x) = \cot x \)**

Domain: \( \{ x \mid x \neq n\pi, \text{ where } n \text{ is an integer} \} \)  \hspace{1cm} Range: \( (-\infty, \infty) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>undefined</td>
</tr>
</tbody>
</table>

\( f(x) = \cot x, 0 < x < \pi \)

(continued)
6.4 Graphs of the Other Circular Functions

- The graph is discontinuous at values of $x$ of the form $x = n\pi$ and has vertical asymptotes at these values.
- Its $x$-intercepts are of the form $x = (2n + 1)\frac{\pi}{2}$.
- Its period is $\pi$.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all $x$ in the domain, $\cot(-x) = -\cot x$.

The tangent function can be graphed directly with a graphing calculator, using the tangent key. To graph the cotangent function, however, we must use one of the identities $\cot x = \frac{1}{\tan x}$ or $\cot x = \frac{\cos x}{\sin x}$ since graphing calculators generally do not have cotangent keys.

Guidelines for Sketching Graphs of Tangent and Cotangent Functions

To graph $y = a \tan bx$ or $y = a \cot bx$, with $b > 0$, follow these steps.

**Step 1** Determine the period, $\frac{\pi}{b}$: To locate two adjacent vertical asymptotes, solve the following equations for $x$:
- For $y = a \tan bx$: $bx = -\frac{\pi}{2}$ and $bx = \frac{\pi}{2}$.
- For $y = a \cot bx$: $bx = 0$ and $bx = \pi$.

**Step 2** Sketch the two vertical asymptotes found in Step 1.

**Step 3** Divide the interval formed by the vertical asymptotes into four equal parts.

**Step 4** Evaluate the function for the first-quarter point, midpoint, and third-quarter point, using the $x$-values found in Step 3.

**Step 5** Join the points with a smooth curve, approaching the vertical asymptotes. Indicate additional asymptotes and periods of the graph as necessary.

EXAMPLE 3 Graphing $y = \tan bx$

Graph $y = \tan 2x$.

**Solution**

**Step 1** The period of this function is $\frac{\pi}{2}$. To locate two adjacent vertical asymptotes, solve $2x = -\frac{\pi}{2}$ and $2x = \frac{\pi}{2}$ (since this is a tangent function). The two asymptotes have equations $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$.

**Step 2** Sketch the two vertical asymptotes $x = \pm\frac{\pi}{4}$, as shown in Figure 47 on the next page.
Step 3 Divide the interval \((-\frac{\pi}{4}, \frac{\pi}{4})\) into four equal parts. This gives the following key \(x\)-values.

first-quarter value: \(-\frac{\pi}{8}\),  
middle value: 0,  
third-quarter value: \(\frac{\pi}{8}\)

Step 4 Evaluate the function for the \(x\)-values found in Step 3.

\[
\begin{array}{ccc}
2x & -\frac{\pi}{4} & 0 & \frac{\pi}{4} \\
\tan 2x & -1 & 0 & 1
\end{array}
\]

Step 5 Join these points with a smooth curve, approaching the vertical asymptotes. See Figure 47. Another period has been graphed, one half period to the left and one half period to the right.

Now try Exercise 21.

**EXAMPLE 4** Graphing \(y = a \tan bx\)

Graph \(y = -3 \tan \frac{1}{2}x\).

**Solution** The period is \(\frac{\pi}{\frac{1}{2}} = 2\pi\). Adjacent asymptotes are at \(x = -\pi\) and \(x = \pi\). Dividing the interval \(-\pi < x < \pi\) into four equal parts gives key \(x\)-values of \(-\frac{\pi}{2}\), 0, and \(\frac{\pi}{2}\). Evaluating the function at these \(x\)-values gives the key points:

\[\left(\frac{-\pi}{2}, 3\right), \quad (0, 0), \quad \left(\frac{\pi}{2}, -3\right)\]

By plotting these points and joining them with a smooth curve, we obtain the graph shown in Figure 48. Because the coefficient \(-3\) is negative, the graph is reflected across the \(x\)-axis compared to the graph of \(y = 3 \tan \frac{1}{2}x\).

Now try Exercise 29.

**NOTE** The function defined by \(y = -3 \tan \frac{1}{2}x\) in Example 4, graphed in Figure 48, has a graph that compares to the graph of \(y = \tan x\) as follows.

1. The period is larger because \(b = \frac{1}{2}\), and \(\frac{1}{2} < 1\).
2. The graph is “stretched” because \(a = -3\), and \(|-3| > 1\).
3. Each branch of the graph goes down from left to right (that is, the function decreases) between each pair of adjacent asymptotes because \(a = -3 < 0\). When \(a < 0\), the graph is reflected across the \(x\)-axis compared to the graph of \(y = |a| \tan bx\).
6.4 Graphs of the Other Circular Functions

EXAMPLE 5  Graphing \( y = a \cot bx \)

Graph \( y = \frac{1}{2} \cot 2x \).

**Solution**  Because this function involves the cotangent, we can locate two adjacent asymptotes by solving the equations \( 2x = 0 \) and \( 2x = \pi \). The lines \( x = 0 \) (the \( y \)-axis) and \( x = \frac{\pi}{2} \) are two such asymptotes. Divide the interval \( (0, \pi) \) into four equal parts, getting key \( x \)-values of \( \frac{\pi}{8}, \frac{\pi}{4}, \) and \( \frac{3\pi}{8} \). Evaluating the function at these \( x \)-values gives the following key points.

\[
\left( \frac{\pi}{8}, \frac{1}{2} \right), \quad \left( \frac{\pi}{4}, 0 \right), \quad \left( \frac{3\pi}{8}, -\frac{1}{2} \right)
\]

Joining these points with a smooth curve approaching the asymptotes gives the graph shown in Figure 49.

Now try Exercise 31.

Like the other circular functions, the graphs of the tangent and cotangent functions may be translated horizontally and vertically.

EXAMPLE 6  Graphing a Tangent Function with a Vertical Translation

Graph \( y = 2 + \tan x \).

**Analytic Solution**  Every value of \( y \) for this function will be 2 units more than the corresponding value of \( y \) in \( y = \tan x \), causing the graph of \( y = 2 + \tan x \) to be translated 2 units up compared with the graph of \( y = \tan x \). See Figure 50.

**Graphing Calculator Solution**  To see the vertical translation, observe the coordinates displayed at the bottoms of the screens in Figures 51 and 52. For \( X = \frac{\pi}{4} = .78539816 \),

\[
Y_1 = \tan X = 1,
\]

while for the same \( X \)-value,

\[
Y_2 = 2 + \tan X = 2 + 1 = 3.
\]

Now try Exercise 37.
EXAMPLE 7  Graphing a Cotangent Function with Vertical and Horizontal Translations

Graph \( y = -2 - \cot \left( x - \frac{\pi}{4} \right) \).

**Solution**  Here \( b = 1 \), so the period is \( \pi \). The graph will be translated down 2 units (because \( c = -2 \)), reflected across the \( x \)-axis (because of the negative sign in front of the cotangent), and will have a phase shift (horizontal translation) \( \frac{\pi}{4} \) unit to the right (because of the argument \( x - \frac{\pi}{4} \)). To locate adjacent asymptotes, since this function involves the cotangent, we solve the following equations:

\[
\begin{align*}
x - \frac{\pi}{4} &= 0, & \text{so } x &= \frac{\pi}{4} \\
& & \text{and } x - \frac{\pi}{4} &= \pi, & \text{so } x &= \frac{5\pi}{4}.
\end{align*}
\]

Dividing the interval \( \frac{\pi}{4} < x < \frac{5\pi}{4} \) into four equal parts and evaluating the function at the three key \( x \)-values within the interval gives these points:

\[
\left( \frac{\pi}{2}, -3 \right), \quad \left( \frac{3\pi}{4}, -2 \right), \quad (\pi, -1)
\]

Join these points with a smooth curve. This period of the graph, along with the one in the domain interval \( \left( -\frac{3\pi}{4}, \frac{\pi}{4} \right) \), is shown in Figure 53.

Now try Exercise 45.

**Addition of Ordinates**  New functions can be formed by adding or subtracting other functions. A function formed by combining two other functions, such as

\[ y = \cos x + \sin x, \]

has historically been graphed using a method known as **addition of ordinates**. (The \( x \)-value of a point is sometimes called its **abscissa**, while its \( y \)-value is called its **ordinate**.) To apply this method to this function, we graph the functions \( y = \cos x \) and \( y = \sin x \). Then, for selected values of \( x \), we add \( \cos x \) and \( \sin x \), and plot the points \( (x, \cos x + \sin x) \). Joining the resulting points with a sinusoidal curve gives the graph of the desired function. While this method illustrates some valuable concepts involving the arithmetic of functions, it is time-consuming.

With graphing calculators, this technique is easily illustrated. Let \( Y_1 = \cos X \), \( Y_2 = \sin X \), and \( Y_3 = Y_1 + Y_2 \). Figure 54 shows the result when \( Y_1 \) and \( Y_2 \) are graphed in thin graph style, and \( Y_3 = \cos X + \sin X \) is graphed in thick graph style. Notice that for \( X = \frac{\pi}{6} \approx .52359878 \), \( Y_1 + Y_2 = Y_3 \).

Now try Exercise 61.
6.4 Exercises

Concept Check In Exercises 1–6, match each function with its graph from choices A–F.

1. \(y = -\csc x\)
2. \(y = -\sec x\)
3. \(y = -\tan x\)
4. \(y = -\cot x\)
5. \(y = \tan \left( x - \frac{\pi}{4} \right) \)
6. \(y = \cot \left( x - \frac{\pi}{4} \right) \)

Graph each function over a one-period interval. See Examples 1 and 2.

7. \(y = 3 \sec \frac{1}{2}x\)
8. \(y = -2 \sec \frac{1}{2}x\)
9. \(-\frac{1}{2} \csc \left( x + \frac{\pi}{2} \right)\)
10. \(y = \frac{1}{2} \csc \left( x - \frac{\pi}{2} \right)\)
11. \(y = \csc \left( x - \frac{\pi}{4} \right)\)
12. \(y = \sec \left( x + \frac{3\pi}{4} \right)\)
13. \(y = \sec \left( x + \frac{\pi}{4} \right)\)
14. \(y = \csc \left( x + \frac{\pi}{3} \right)\)
15. \(y = \csc \left( \frac{1}{2}x + \frac{\pi}{3} \right)\)
16. \(y = \csc \left( \frac{1}{2}x - \frac{\pi}{4} \right)\)
17. \(y = 2 + 3 \sec(2x - \pi)\)
18. \(y = 1 - 2 \csc \left( x + \frac{\pi}{2} \right)\)
19. \(y = 1 - \frac{1}{2} \csc \left( x - \frac{3\pi}{4} \right)\)
20. \(y = 2 + \frac{1}{4} \sec \left( \frac{1}{2}x - \pi \right)\)

Graph each function over a one-period interval. See Examples 3–5.

21. \(y = \tan 4x\)
22. \(y = \tan \frac{1}{2}x\)
23. \(y = 2 \tan x\)
24. \(y = 2 \cot x\)
25. \(y = 2 \tan \frac{1}{4}x\)
26. \(y = \frac{1}{2} \cot x\)
27. \(y = \cot 3x\)
28. \(y = -\cot \frac{1}{2}x\)
29. \(y = -2 \tan \frac{1}{4}x\)
30. \(y = 3 \tan \frac{1}{2}x\)
31. \(y = \frac{1}{2} \cot 4x\)
32. \(-\frac{1}{2} \cot 2x\)

Graph each function over a two-period interval. See Examples 6 and 7.

33. \(y = \tan(2x - \pi)\)
34. \(y = \tan \left( \frac{x}{2} + \pi \right)\)
35. \(y = \cot \left( 3x + \frac{\pi}{4} \right)\)
36. \( y = \cot \left( 2x - \frac{3\pi}{2} \right) \)  
37. \( y = 1 + \tan x \)  
38. \( y = 1 - \tan x \)  
39. \( y = 1 - \cot x \)  
40. \( y = -2 - \cot x \)  
41. \( y = -1 + 2 \tan x \)  
42. \( y = 3 + \frac{1}{2} \tan x \)  
43. \( y = -1 + \frac{1}{2} \cot(2x - 3\pi) \)  
44. \( y = -2 + 3 \tan(4x + \pi) \)  
45. \( y = 1 - 2 \cot \left( x + \frac{\pi}{2} \right) \)  
46. \( y = \frac{2}{3} \tan \left( \frac{3}{4}x - \pi \right) - 2 \)

Concept Check In Exercises 47–52, tell whether each statement is true or false. If false, tell why.

47. The smallest positive number \( k \) for which \( x = k \) is an asymptote for the tangent function is \( \frac{\pi}{2} \).

48. The smallest positive number \( k \) for which \( x = k \) is an asymptote for the cotangent function is \( \frac{\pi}{2} \).

49. The tangent and secant functions are undefined for the same values.

50. The secant and cosecant functions are undefined for the same values.

51. The graph of \( y = \tan x \) in Figure 45 suggests that \( \tan(-x) = \tan x \) for all \( x \) in the domain of \( \tan x \).

52. The graph of \( y = \sec x \) in Figure 40 suggests that \( \sec(-x) = \sec x \) for all \( x \) in the domain of \( \sec x \).

53. Concept Check If \( c \) is any number, then how many solutions does the equation \( c = \tan x \) have in the interval \((-2\pi, 2\pi)\)?

54. Concept Check If \( c \) is any number such that \(-1 < c < 1\), then how many solutions does the equation \( c = \sec x \) have over the entire domain of the secant function?

55. Consider the function defined by \( f(x) = -4 \tan(2x + \pi) \). What is the domain of \( f \)? What is its range?

56. Consider the function defined by \( g(x) = -2 \csc(4x + \pi) \). What is the domain of \( g \)? What is its range?

(Modeling) Solve each problem.

57. Distance of a Rotating Beacon A rotating beacon is located at point \( A \) next to a long wall. (See the figure.) The beacon is 4 m from the wall. The distance \( d \) is given by

\[ d = 4 \tan 2\pi t, \]

where \( t \) is time measured in seconds since the beacon started rotating. (When \( t = 0 \), the beacon is aimed at point \( R \). When the beacon is aimed to the right of \( R \), the value of \( d \) is positive; \( d \) is negative if the beacon is aimed to the left of \( R \).) Find \( d \) for each time.

(a) \( t = 0 \)
(b) \( t = .4 \)
(c) \( t = .8 \)
(d) \( t = 1.2 \)
(e) Why is \( .25 \) a meaningless value for \( t \)?
58. **Distance of a Rotating Beacon** In the figure for Exercise 57, the distance \( a \) is given by

\[
a = 4|\sec 2\pi t|.
\]

Find \( a \) for each time.

(a) \( t = 0 \)  
(b) \( t = 0.86 \)  
(c) \( t = 1.24 \)

59. Simultaneously graph \( y = \tan x \) and \( y = x \) in the window \([-1, 1] \times [-1, 1]\) with a graphing calculator. Write a sentence or two describing the relationship of \( \tan x \) and \( x \) for small \( x \)-values.

60. Between each pair of successive asymptotes, a portion of the graph of \( y = \sec x \) or \( y = \csc x \) resembles a parabola. Can each of these portions actually be a parabola? Explain.

Use a graphing calculator to graph \( Y_1, Y_2, \) and \( Y_3 = Y_1 + Y_2 \) on the same screen. Evaluate each of the three functions at \( X = \frac{\pi}{6} \), and verify that \( Y_1(\frac{\pi}{6}) + Y_2(\frac{\pi}{6}) = Y_3(\frac{\pi}{6}) \). See the discussion on addition of ordinates.

61. \( Y_1 = \sin X \), \( Y_2 = \sin 2X \)

62. \( Y_1 = \cos X \), \( Y_2 = \sec X \)

---

**Relating Concepts**

*For individual or collaborative investigation*  
*(Exercises 63–68)*

Consider the function defined by \( y = -2 - \cot(x - \frac{\pi}{4}) \) from Example 7. Work these exercises in order.

63. What is the smallest positive number for which \( y = \cot x \) is undefined?

64. Let \( k \) represent the number you found in Exercise 63. Set \( x - \frac{\pi}{4} \) equal to \( k \), and solve to find the smallest positive number for which \( \cot(x - \frac{\pi}{4}) \) is undefined.

65. Based on your answer in Exercise 64 and the fact that the cotangent function has period \( \pi \), give the general form of the equations of the asymptotes of the graph of \( y = -2 - \cot(x - \frac{\pi}{4}) \). Let \( n \) represent any integer.

66. Use the capabilities of your calculator to find the smallest positive \( x \)-intercept of the graph of this function.

67. Use the fact that the period of this function is \( \pi \) to find the next positive \( x \)-intercept.

68. Give the solution set of the equation \(-2 - \cot(x - \frac{\pi}{4}) = 0\) over all real numbers. Let \( n \) represent any integer.

55. domain: \( \left\{ x \mid x \neq (2n + 1)\frac{\pi}{4} \right\} \), range: \((-\infty, \infty) \)

56. domain: \( \left\{ x \mid x \neq \frac{n\pi}{4} \right\} \), range: \((-\infty, -2] \cup [2, \infty) \)

57. (a) 0 m (b) -2.9 m (c) -12.3 m (d) 12.3 m (e) It leads to \( \tan \frac{\pi}{2} \), which is undefined.

58. (a) 4 m (b) 6.3 m (c) 63.7 m

In Exercises 61 and 62, we show the display for \( Y_1 + Y_2 \) at \( X = \frac{\pi}{6} \).

61. ![Graph](image1)

62. ![Graph](image2)
6.5 Harmonic Motion

**Simple Harmonic Motion**  In part A of Figure 55, a spring with a weight attached to its free end is in equilibrium (or rest) position. If the weight is pulled down $a$ units and released (part B of the figure), the spring’s elasticity causes the weight to rise $a$ units above the equilibrium position, as seen in part C, and then oscillate about the equilibrium position. If friction is neglected, this oscillatory motion is described mathematically by a sinusoid. Other applications of this type of motion include sound, electric current, and electromagnetic waves.

To develop a general equation for such motion, consider Figure 56. Suppose the point $P(x, y)$ moves around the circle counterclockwise at a uniform angular speed $\omega$. Assume that at time $t = 0$, $P$ is at $(a, 0)$. The angle swept out by ray $OP$ at time $t$ is given by $\theta = \omega t$. The coordinates of point $P$ at time $t$ are

$$x = a \cos \theta = a \cos \omega t \quad \text{and} \quad y = a \sin \theta = a \sin \omega t.$$

As $P$ moves around the circle from the point $(a,0)$, the point $Q(0, y)$ oscillates back and forth along the $y$-axis between the points $(0,a)$ and $(0,-a)$. Similarly, the point $R(x,0)$ oscillates back and forth between $(a,0)$ and $(-a,0)$. This oscillatory motion is called **simple harmonic motion**.
The amplitude of the motion is $|a|$, and the period is $\frac{2\pi}{\omega}$. The moving points $P$ and $Q$ or $P$ and $R$ complete one oscillation or cycle per period. The number of cycles per unit of time, called the frequency, is the reciprocal of the period, $\frac{\omega}{2\pi}$, where $\omega > 0$.

### Simple Harmonic Motion

The position of a point oscillating about an equilibrium position at time $t$ is modeled by either

$$s(t) = a \cos \omega t \quad \text{or} \quad s(t) = a \sin \omega t,$$

where $a$ and $\omega$ are constants, with $\omega > 0$. The amplitude of the motion is $|a|$, the period is $\frac{2\pi}{\omega}$, and the frequency is $\frac{\omega}{2\pi}$.

### EXAMPLE 1  Modeling the Motion of a Spring

Suppose that an object is attached to a coiled spring such as the one in Figure 55. It is pulled down a distance of 5 in. from its equilibrium position, and then released. The time for one complete oscillation is 4 sec.

(a) Give an equation that models the position of the object at time $t$.

(b) Determine the position at $t = 1.5$ sec.

(c) Find the frequency.

**Solution**

(a) When the object is released at $t = 0$, the distance of the object from the equilibrium position is 5 in. below equilibrium. If $s(t)$ is to model the motion, then $s(0)$ must equal $-5$. We use

$$s(t) = a \cos \omega t,$$

with $a = -5$. We choose the cosine function because $\cos \omega (0) = \cos 0 = 1$, and $-5 \cdot 1 = -5$. (Had we chosen the sine function, a phase shift would have been required.) The period is 4, so

$$\frac{2\pi}{\omega} = 4, \quad \text{or} \quad \omega = \frac{\pi}{2}. \quad \text{Solve for } \omega. \quad \text{(Section 1.1)}$$

Thus, the motion is modeled by

$$s(t) = -5 \cos \frac{\pi}{2} t.$$

(b) After 1.5 sec, the position is

$$s(1.5) = -5 \cos \left[ \frac{\pi}{2} (1.5) \right] = 3.54 \text{ in}.$$  

Since $3.54 > 0$, the object is above the equilibrium position.

(c) The frequency is the reciprocal of the period, or $\frac{1}{4}$.

Now try Exercise 9.
EXAMPLE 2 Analyzing Harmonic Motion

Suppose that an object oscillates according to the model

\[ s(t) = 8 \sin 3t, \]

where \( t \) is in seconds and \( s(t) \) is in feet. Analyze the motion.

Solution The motion is harmonic because the model is of the form \( s(t) = a \sin \omega t \). Because \( a = 8 \), the object oscillates 8 ft in either direction from its starting point. The period \( \frac{2\pi}{3} = 2.1 \) is the time, in seconds, it takes for one complete oscillation. The frequency is the reciprocal of the period, so the object completes \( \frac{3}{\pi} \approx .48 \) oscillation per sec.

Now try Exercise 15.

Damped Oscillatory Motion In the example of the stretched spring, we disregard the effect of friction. Friction causes the amplitude of the motion to diminish gradually until the weight comes to rest. In this situation, we say that the motion has been *damped* by the force of friction. Most oscillatory motions are damped, and the decrease in amplitude follows the pattern of exponential decay. A typical example of damped oscillatory motion is provided by the function defined by

\[ s(t) = e^{-t} \sin t. \]

Figure 57 shows how the graph of \( y_3 = e^{-x} \sin x \) is bounded above by the graph of \( y_1 = e^{-x} \) and below by the graph of \( y_2 = -e^{-x} \). The damped motion curve dips below the \( x \)-axis at \( x = \pi \) but stays above the graph of \( y_2 \). Figure 58 shows a traditional graph of \( s(t) = e^{-t} \sin t \), along with the graph of \( y = \sin t \).

Now try Exercise 21.

Shock absorbers are put on an automobile in order to damp oscillatory motion. Instead of oscillating up and down for a long while after hitting a bump or pothole, the oscillations of the car are quickly damped out for a smoother ride.
6.5 Exercises

1. (a) \( s(t) = 2 \cos 4\pi t \)  
   (b) \( s(1) = 2; \) The weight is neither moving upward nor downward. At \( t = 1, \) the motion of the weight is changing from up to down.

2. (a) \( s(t) = 5 \cos \frac{4\pi}{3} t \)  
   (b) \( s(1) = -2.5; \) upward

3. (a) \( s(t) = -3 \cos 2.5\pi t \)  
   (b) \( s(1) = 0; \) upward

4. (a) \( s(t) = -4 \cos \frac{5\pi}{3} t \)  
   (b) \( s(1) = -2; \) downward

5. \( s(t) = 0.21 \cos 55\pi t \)

6. \( s(t) = 0.11 \cos 220\pi t \)

7. \( s(t) = 0.14 \cos 110\pi t \)

8. \( s(t) = 0.06 \cos 440\pi t \)

9. (a) \( s(t) = -4 \cos \frac{2\pi}{3} t \)  
   (b) 3.46 in. (c) \( \frac{1}{3} \) 

(Modeling) Springs  
Suppose that a weight on a spring has initial position \( s(0) \) and period \( P. \)

(a) Find a function \( s \) given by \( s(t) = a \cos \omega t \) that models the displacement of the weight.

(b) Evaluate \( s(1). \) Is the weight moving upward, downward, or neither when \( t = 1? \)

1. \( s(0) = 2 \text{ in.; } P = 0.5 \text{ sec} \)  
   2. \( s(0) = 5 \text{ in.; } P = 1.5 \text{ sec} \)  
   3. \( s(0) = -3 \text{ in.; } P = 0.8 \text{ sec} \)  
   4. \( s(0) = -4 \text{ in.; } P = 1.2 \text{ sec} \)

(Modeling) Music  
A note on the piano has given frequency \( F. \) Suppose the maximum displacement at the center of the piano wire is given by \( s(0). \) Find constants \( a \) and \( \omega \) so that the equation \( s(t) = a \cos \omega t \) models this displacement. Graph \( s \) in the viewing window \([0,0.05] \) by \([-3,3].\)

5. \( F = 27.5; s(0) = 0.21 \)  
   6. \( F = 110; s(0) = 0.11 \)  
   7. \( F = 55; s(0) = 0.14 \)  
   8. \( F = 220; s(0) = 0.06 \)

(Modeling) Spring  
Solve each problem. See Examples 1 and 2.

9. Spring  
An object is attached to a coiled spring, as in Figure 55. It is pulled down a distance of 4 units from its equilibrium position, and then released. The time for one complete oscillation is 3 sec.

(a) Give an equation that models the position of the object at time \( t. \)

(b) Determine the position at \( t = 1.25 \) sec.

(c) Find the frequency.

10. Spring  
Repeat Exercise 9, but assume that the object is pulled down 6 units and the time for one complete oscillation is 4 sec.

11. Particle Movement  
Write the equation and then determine the amplitude, period, and frequency of the simple harmonic motion of a particle moving uniformly around a circle of radius 2 units, with angular speed

(a) 2 radians per sec  
(b) 4 radians per sec.

12. Pendulum  
What are the period \( P \) and frequency \( T \) of oscillation of a pendulum of length \( \frac{1}{2} \text{ ft}. \) \((\text{Hint}: P = 2\pi \sqrt{\frac{L}{g}}, \text{ where } L \text{ is the length of the pendulum in feet and } P \text{ is in seconds.})\)

13. Pendulum  
In Exercise 12, how long should the pendulum be to have period 1 sec?

14. Spring  
The formula for the up and down motion of a weight on a spring is given by

\[ s(t) = a \sin \sqrt{\frac{k}{m}} t. \]

If the spring constant \( k \) is 4, what mass \( m \) must be used to produce a period of 1 sec?

15. Spring  
The height attained by a weight attached to a spring set in motion is

\[ s(t) = -4 \cos 8\pi t \]

inches after \( t \) seconds.

(a) Find the maximum height that the weight rises above the equilibrium position of \( y = 0. \)

(b) When does the weight first reach its maximum height, if \( t \geq 0? \)

(c) What are the frequency and period?
10. (a) \( s(t) = -6 \cos \frac{\pi}{2} t \)
(b) 2.30 in. (c) \( \frac{1}{4} \)

11. (a) \( s(t) = 2 \sin 2t \); 
   amplitude: 2; period: \( \pi \); 
   frequency: \( \frac{1}{\pi} \)
(b) \( s(t) = 2 \sin 4t \); amplitude: 2; 
   period: \( \frac{\pi}{2} \); frequency: \( \frac{2}{\pi} \)

12. period: \( \frac{\pi}{4} \); frequency: \( \frac{4}{\pi} \)

13. \( \frac{8}{\pi^2} \) 14. \( \frac{1}{\pi^2} \)

15. (a) 4 in. (b) after \( \frac{1}{8} \) sec 
   (c) 4 cycles per sec; \( \frac{1}{4} \) sec 

16. (a) amplitude: \( \frac{1}{2} \); 
   period: \( \sqrt{2}\pi \); frequency: \( \frac{\sqrt{2}}{2\pi} \)
(b) \( s(t) = \frac{1}{2} \sin \sqrt{2} t \)

17. (a) 5 in. (b) 2 cycles per sec; 
   \( \frac{1}{2} \) sec (c) after \( \frac{1}{4} \) sec 
   (d) approximately 4; After 1.3 sec, the weight is about 4 in. 
   above the equilibrium position.

18. (a) 4 in. (b) \( \frac{5}{\pi} \) cycles 
   per sec; \( \frac{\pi}{5} \) sec 
   (c) after \( \frac{\pi}{10} \) sec 
   (d) approximately 2; After 1.466 sec, the weight is about 2 in. 
   above the equilibrium position.

19. (a) \( s(t) = -3 \cos 12t \)
(b) \( \frac{\pi}{6} \) sec 

20. (a) \( s(t) = -2 \cos 6\pi t \)
(b) 3 cycles per sec 21. 0; \( \pi \), 
   They are the same. 22. for \( y_1 \) and \( y_2 \): 
   \( 1.5707963 = \frac{\pi}{2} \); for \( y_1 \) and \( y_3 \): none in \([0, \pi] \); 
   Because \( \sin \frac{\pi}{2} = 1 \), 
   \( e^{-v/2} \sin \frac{\pi}{2} = e^{-v/2} \).

22. Find any points of intersection of \( y_1 \) and \( y_2 \) or \( y_1 \) and \( y_3 \). How are these points related to the graph of \( y = \sin x \)?

21. Find the \( t \)-intercepts of the graph of \( y_1 \). Explain the relationship of these intercepts with the \( x \)-intercepts of the graph of \( y = \sin x \).
Chapter 6 Summary

KEY TERMS

6.1 radian
sector of a circle

6.2 unit circle
unit circle

6.3 linear speed \( v \)
period

6.4 periodic function
point

6.5 sine wave (sinusoid)

phase shift

argument

amplitude

frequency
damped oscillatory

motion

QUICK REVIEW

CONCEPTS

EXAMPLES

6.1 Radian Measure

An angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of 1 radian.

Degree/Radian Relationship \( 180^\circ = \pi \) radians

Converting Between Degrees and Radians

1. Multiply a degree measure by \( \frac{\pi}{180} \) radian and simplify to convert to radians.

2. Multiply a radian measure by \( \frac{180}{\pi} \) and simplify to convert to degrees.

Arc Length

The length \( s \) of the arc intercepted on a circle of radius \( r \) by a central angle of measure \( \theta \) radians is given by the product of the radius and the radian measure of the angle, or

\[ s = r\theta, \quad \theta \text{ in radians.} \]

Area of a Sector

The area of a sector of a circle of radius \( r \) and central angle \( \theta \) is given by

\[ A = \frac{1}{2} r^2 \theta, \quad \theta \text{ in radians.} \]

Convert 135° to radians.

\[ 135^\circ = 135 \left( \frac{\pi}{180} \right) = \frac{3\pi}{4} \text{ radians} \]

Convert \(-\frac{5\pi}{3}\) radians to degrees.

\[ -\frac{5\pi}{3} \text{ radians} = -\frac{5\pi}{3} \left( \frac{180^\circ}{\pi} \right) = -300^\circ \]

In the figure, \( s = r\theta \) so

\[ \theta = \frac{s}{r} = \frac{3}{4} \text{ radian.} \]

The area of the sector in the figure is

\[ A = \frac{1}{2} (4^2) \left( \frac{3}{4} \right) = 6 \text{ square units.} \]
6.2 The Unit Circle and Circular Functions

Circular Functions
Start at the point (1, 0) on the unit circle \( x^2 + y^2 = 1 \) and lay off an arc of length \(|s|\) along the circle, going counterclockwise if \( s \) is positive, and clockwise if \( s \) is negative. Let the endpoint of the arc be at the point \((x, y)\). The six circular functions of \( s \) are defined as follows. (Assume that no denominators are 0.)

\[
\begin{align*}
\sin s &= y \\
\cos s &= x \\
\tan s &= \frac{y}{x} \\
\csc s &= \frac{1}{y} \\
\sec s &= \frac{1}{x} \\
\cot s &= \frac{x}{y}
\end{align*}
\]

The Unit Circle

Formulas for Angular and Linear Speed

<table>
<thead>
<tr>
<th>Angular Speed</th>
<th>Linear Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega = \frac{\theta}{t} )</td>
<td>( v = \frac{s}{t} )</td>
</tr>
<tr>
<td>( \omega ) in radians per unit time, ( \theta ) in radians</td>
<td>( v = \frac{r \theta}{t} )</td>
</tr>
<tr>
<td>( v = \frac{r \omega}{t} )</td>
<td>( v = r \omega )</td>
</tr>
</tbody>
</table>

A belt runs a pulley of radius 8 in. at 60 revolutions per min. Find the angular speed \( \omega \) in radians per minute, and the linear speed \( v \) of the belt in inches per minute.

\[
\omega = 60(2\pi) = 120\pi \text{ radians per min}
\]

\[
v = r\omega = 8(120\pi) = 960\pi \text{ in. per min}
\]

Use the unit circle to find each value.

\[
\begin{align*}
\sin \frac{5\pi}{6} &= \frac{1}{2} \\
\cos \frac{3\pi}{2} &= 0 \\
\tan \frac{\pi}{4} &= \frac{\sqrt{2}}{\sqrt{2}} = 1 \\
\csc \frac{7\pi}{4} &= \frac{1}{-\sqrt{2}} = -\sqrt{2} \\
\sec \frac{7\pi}{6} &= \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3} \\
\cot \frac{\pi}{3} &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}
\end{align*}
\]
6.3 Graphs of the Sine and Cosine Functions

Sine and Cosine Functions

Graph $y = \sin 3x$.

Graph $y = \cos x$.

Graph $y = -\cos x$.

The graph of

$$y = c + a \sin (x - d) \quad \text{or} \quad y = c + a \cos (x - d),$$

$b > 0$, has

1. amplitude $|a|$,  
2. period $\frac{2\pi}{b}$,  
3. vertical translation $c$ units up if $c > 0$ or $|c|$ units down if $c < 0$, and
4. phase shift $d$ units to the right if $d > 0$ or $|d|$ units to the left if $d < 0$.

See pages 562 and 565 for a summary of graphing techniques.

6.4 Graphs of the Other Circular Functions

Cosecant and Secant Functions

Graph one period of $y = \sec \left(x + \frac{\pi}{4}\right)$.

Graph $y = \csc x$.

Graph $y = \sec x$.

See page 577 for a summary of graphing techniques.
Chapter 6 Review Exercises

1. Consider each angle in standard position having the given radian measure. In what quadrant does the terminal side lie?
   (a) \( \frac{3\pi}{4} \) (b) \( \frac{40\pi}{9} \) (c) \(-2\) (d) \(7\)

2. Which is larger—an angle of 1° or an angle of 1 radian? Discuss and justify your answer.

Convert each degree measure to radians. Leave answers as multiples of \( \pi \).

3. 120°

Convert each radian measure to degrees.

4. 800°

5. \( \frac{5\pi}{4} \)

6. \( -\frac{6\pi}{5} \)
7. **Railroad Engineering** The term *grade* has several different meanings in construction work. Some engineers use the term grade to represent \( \frac{1}{100} \) of a right angle and express grade as a percent. For instance, an angle of \( .9^\circ \) would be referred to as a 1\% grade. (*Source: Hay, W., *Railroad Engineering*, John Wiley & Sons, 1982.*

(a) By what number should you multiply a grade to convert it to radians?

(b) In a rapid-transit rail system, the maximum grade allowed between two stations is 3.5\%. Express this angle in degrees and radians.

8. **Concept Check** Suppose the tip of the minute hand of a clock is 2 in. from the center of the clock. For each of the following durations, determine the distance traveled by the tip of the minute hand.

(a) 20 min

(b) 3 hr

Solve each problem. Use a calculator as necessary.

9. **Arc Length** The radius of a circle is 15.2 cm. Find the length of an arc of the circle intercepted by a central angle of \( \frac{3\pi}{2} \) radians.

10. **Area of a Sector** A central angle of \( \frac{2\pi}{3} \) radians forms a sector of a circle. Find the area of the sector if the radius of the circle is 28.69 in.

11. **Height of a Tree** A tree 2000 yd away subtends an angle of \( 1^\circ \ 10' \). Find the height of the tree to two significant digits.

12. **Rotation of a Seesaw** The seesaw at a playground is 12 ft long. Through what angle does the board rotate when a child rises 3 ft along the circular arc?

*Consider the figure here for Exercises 13 and 14.*

13. What is the measure of \( \theta \) in radians?

14. What is the area of the sector?

Solve each problem.

15. Find the time \( t \) if \( \theta = \frac{5\pi}{12} \) radians and \( \omega = \frac{8\pi}{9} \) radians per sec.

16. Find angle \( \theta \) if \( t = 12 \) sec and \( \omega = 9 \) radians per sec.

17. **Linear Speed of a Flywheel** Find the linear speed of a point on the edge of a flywheel of radius 7 m if the flywheel is rotating 90 times per sec.

18. **Angular Speed of a Ferris Wheel** A Ferris wheel has radius 25 ft. If it takes 30 sec for the wheel to turn \( \frac{5\pi}{3} \) radians, what is the angular speed of the wheel?
19. **Concept Check** Consider the area-of-a-sector formula \( A = \frac{1}{2} r^2 \theta \). What well-known formula corresponds to the special case \( \theta = 2\pi \)?

Find the exact function value. Do not use a calculator.

- 20. \( \cos \left( \frac{2\pi}{3} \right) \)
- 21. \( \tan \left( \frac{-7\pi}{3} \right) \)
- 22. \( \csc \left( \frac{-11\pi}{6} \right) \)

Use a calculator to find an approximation for each circular function value. Be sure your calculator is set in radian mode.

- 23. \( \cos(-.2443) \)
- 24. \( \cot 3.0543 \)

25. Approximate the value of \( s \) in the interval \([0, \frac{\pi}{2}]\) if \( \sin s = .4924 \).

Find the exact value of \( s \) in the given interval that has the given circular function value. Do not use a calculator.

- 26. \( \left[ \frac{\pi}{2}, \pi \right] \); \( \tan s = -\sqrt{3} \)
- 27. \( \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \); \( \sec s = -\frac{2\sqrt{3}}{3} \)

**Modeling** Solve each problem.

28. **Phase Angle of the Moon** Because the moon orbits Earth, we observe different phases of the moon during the period of a month. In the figure, \( t \) is called the phase angle.

![Diagram](image)

The phase \( F \) of the moon is computed by

\[
F(t) = \frac{1}{2} (1 - \cos t),
\]

and gives the fraction of the moon’s face that is illuminated by the sun. (Source: Duffet-Smith, P., *Practical Astronomy with Your Calculator*, Cambridge University Press, 1988.) Evaluate each expression and interpret the result.

- (a) \( F(0) \)
- (b) \( F \left( \frac{\pi}{2} \right) \)
- (c) \( F(\pi) \)
- (d) \( F \left( \frac{3\pi}{2} \right) \)

29. **Atmospheric Effect on Sunlight** The shortest path for the sun’s rays through Earth’s atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of \( \csc \theta \), where \( \theta \) is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reaches Earth’s surface. See the figure at the top of the next page. (Source: Winter, C., R. Sizmann, and Vant-Hunt (Editors), *Solar Power Plants*, Springer-Verlag, 1991.)
29.  (b) \( \frac{\pi}{6} \)  (c) less ultraviolet light when \( \theta = \frac{\pi}{3} \)  
30.  B  
31.  D  
32.  2; 2\( \pi \); none; none  
33.  not applicable; \( \frac{\pi}{3} \); none; none  
34.  \( \frac{1}{2}; \frac{2\pi}{3} \); none; none  
35.  2; \( \frac{2\pi}{3} \); none; none  
36.  2; \( \frac{\pi}{3} \); none; none  
37.  \( \frac{1}{4}; \frac{3\pi}{4} \); 3 up; none  
38.  3; 2\( \pi \); none; \( \frac{\pi}{2} \) to the left  
39.  1; 2\( \pi \); none; \( \frac{3\pi}{4} \) to the right  
40.  not applicable; \( \pi \); none; \( \frac{\pi}{8} \) to the right  
41.  tangent  
42.  sine  
43.  cosine  
44.  

(a) Verify that \( d = h \csc \theta \).  
(b) Determine \( \theta \) when \( d = 2h \).  
(c) The atmosphere filters out the ultraviolet light that causes skin to burn. Compare the difference between sunbathing when \( \theta = \frac{\pi}{2} \) and when \( \theta = \frac{\pi}{3} \). Which measure gives less ultraviolet light?  
30.  Concept Check  Which one of the following is true about the graph of \( y = 4 \sin 2x \)?  
A.  It has amplitude 2 and period \( \frac{\pi}{2} \).  
B.  It has amplitude 4 and period \( \pi \).  
C.  Its range is \([-2, 2]\).  
D.  Its range is \([-4, 0]\).  
31.  Concept Check  Which one of the following is false about the graph of \( y = -3 \cos \frac{1}{2}x \)?  
A.  Its range is \([-3, 3]\).  
B.  Its domain is \((\infty, \infty)\).  
C.  Its amplitude is 3, and its period is \( 4\pi \).  
D.  Its amplitude is 3, and its period is \( \pi \).  

For each function, give the amplitude, period, vertical translation, and phase shift, as applicable.  
32.  \( y = 2 \sin x \)  
33.  \( y = \tan 3x \)  
34.  \( y = -\frac{1}{2} \cos 3x \)  
35.  \( y = 2 \sin 5x \)  
36.  \( y = 1 + 2 \sin \frac{1}{4}x \)  
37.  \( y = 3 - \frac{1}{4} \cos \frac{2}{3}x \)  
38.  \( y = 3 \cos \left(x + \frac{\pi}{2}\right) \)  
39.  \( y = -\sin \left(x - \frac{3\pi}{4}\right) \)  
40.  \( y = \frac{1}{2} \csc \left(2x - \frac{\pi}{4}\right) \)  

Concept Check  Identify the circular function that satisfies each description.  
41.  period is \( \pi \), \( x \)-intercepts are of the form \( n\pi \), where \( n \) is an integer  
42.  period is \( 2\pi \), graph passes through the origin  
43.  period is \( 2\pi \), graph passes through the point \( (\frac{\pi}{2}, 0) \)  

Graph each function over a one-period interval.  
44.  \( y = 3 \cos 2x \)  
45.  \( y = \frac{1}{2} \cot 3x \)  
46.  \( y = \cos \left(x - \frac{\pi}{4}\right) \)  
47.  \( y = \tan \left(x - \frac{\pi}{2}\right) \)  
48.  \( y = 1 + 2 \cos 3x \)  
49.  \( y = -1 - 3 \sin 2x \)  
50.  Explain how by observing the graphs of \( y = \sin x \) and \( y = \cos x \) on the same axes, one can see that for exactly two \( x \)-values in \([0, 2\pi]\), \( \sin x = \cos x \). What are the two \( x \)-values?
51. Viewing Angle to an Object Let a person $h_1$ feet tall stand $d$ feet from an object $h_2$ feet tall, where $h_2 > h_1$. Let $\theta$ be the angle of elevation to the top of the object. See the figure.

(a) Show that $d = (h_2 - h_1) \cot \theta$.
(b) Let $h_2 = 55$ and $h_1 = 5$. Graph $d$ for the interval $0 < \theta \leq \frac{\pi}{2}$.

52. (Modeling) Tides The figure shows a function $f$ that models the tides in feet at Clearwater Beach, Florida, $x$ hours after midnight starting on August 26, 1998. (Source: Pentcheff, D., *WWW Tide and Current Predictor*.)

(a) Find the time between high tides.
(b) What is the difference in water levels between high tide and low tide?
(c) The tides can be modeled by $f(x) = .6 \cos[.511(x - 2.4)] + 2$.

Estimate the tides when $x = 10$.

53. (Modeling) Maximum Temperatures The maximum afternoon temperature in a given city might be modeled by

$$t = 60 - 30 \cos \frac{x\pi}{6},$$

where $t$ represents the maximum afternoon temperature in month $x$, with $x = 0$ representing January, $x = 1$ representing February, and so on. Find the maximum afternoon temperature for each month.

(a) January  (b) April  (c) May  (d) June  (e) August  (f) October

<table>
<thead>
<tr>
<th>Month</th>
<th>°F</th>
<th>Month</th>
<th>°F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>25</td>
<td>July</td>
<td>74</td>
</tr>
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<td>Feb</td>
<td>28</td>
<td>Aug</td>
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<td>61</td>
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</tr>
<tr>
<td>June</td>
<td>72</td>
<td>Dec</td>
<td>28</td>
</tr>
</tbody>
</table>


54. (Modeling) Average Monthly Temperature

The average monthly temperature (in °F) in Chicago, Illinois, is shown in the table.

(a) Plot the average monthly temperature over a two-year period. Let $x = 1$ correspond to January of the first year.
(b) Determine a model function of the form $f(x) = a \sin b(x - d) + c$, where $a$, $b$, $c$, and $d$ are constants.
(c) Explain the significance of each constant.
(d) Graph $f$ together with the data on the same coordinate axes. How well does $f$ model the data?
(e) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data.

55. (Modeling) Pollution Trends The amount of pollution in the air fluctuates with the seasons. It is lower after heavy spring rains and higher after periods of little rain. In addition to this seasonal fluctuation, the long-term trend is upward. An idealized graph of this situation is shown in the figure on the next page.
CHAPTER 6 Test 601

55. (a) 100  (b) 258  (c) 122  (d) 296  56. amplitude: 3; period: π; frequency: \( \frac{1}{\pi} \)
57. amplitude: 4; period: 2; frequency: \( \frac{1}{2} \) 58. The period is the time to complete one cycle. The amplitude is the maximum distance (on either side) from the initial point. 59. The frequency is the number of cycles in one unit of time; \(-4; \ 0; \ -2\sqrt{2} \)

55. (continued) Circular functions can be used to model the fluctuating part of the pollution levels, and exponential functions can be used to model long-term growth. The pollution level in a certain area might be given by

\[ y = 7(1 - \cos 2\pi x)(x + 10) + 100e^{x^2}, \]

where \( x \) is the time in years, with \( x = 0 \) representing January 1 of the base year. July 1 of the same year would be represented by \( x = .5 \). October 1 of the following year would be represented by \( x = 1.75 \), and so on. Find the pollution levels on each date.
(a) January 1, base year  (b) July 1, base year  (c) January 1, following year  (d) July 1, following year

An object in simple harmonic motion has position function \( s \) inches from an initial point, where \( t \) is the time in seconds. Find the amplitude, period, and frequency.

56. \( s(t) = 3 \cos 2t \) 57. \( s(t) = 4 \sin \pi t \)

58. In Exercise 56, what does the period represent? What does the amplitude represent?
59. In Exercise 57, what does the frequency represent? Find the position of the object from the initial point at 1.5 sec, 2 sec, and 3.25 sec.

**Chapter 6 Test**

1. \( \frac{2\pi}{3} \)  2. \( -\frac{\pi}{4} \)  3. \( .09 \)
4. 135°  5. \( -210° \)  6. 229.18°
7. (a) \( \frac{4}{3} \)  (b) 15,000 cm²
8. 2 radians  9. (a) \( \frac{\pi}{3} \) radians  
(b) \( \frac{10\pi}{3} \) cm  (c) \( \frac{5\pi}{9} \) cm per sec

1. 120°  2. \(-45° \)  3. \( 5° \) (to the nearest hundredth)

**Convert each radian measure to degrees.**

4. \( \frac{3\pi}{4} \)  5. \( -\frac{7\pi}{6} \)  6. \( 4 \) (to the nearest hundredth)

7. A central angle of a circle with radius 150 cm cuts off an arc of 200 cm. Find each measure.
   (a) the radian measure of the angle 
   (b) the area of a sector with that central angle

8. **Rotation of Gas Gauge Arrow**  The arrow on a car’s gasoline gauge is \( \frac{1}{2} \) in. long. See the figure. Through what angle does the arrow rotate when it moves 1 in. on the gauge?
10. \( \frac{\sqrt{2}}{2} \)  11. \( -\frac{\sqrt{3}}{2} \)  
12. undefined  13. \(-2\)  
14. (a) 0.9716  (b) \( \frac{\pi}{3} \)  
15. (a) \( \pi \)  (b) 6  (c) \([-3, 9]\)  
   (d) \(-3\)  (e) \( \frac{\pi}{4} \) to the left  
   (that is, \(-\frac{\pi}{4}\))  
16.  17.  
18.  
19.  
20.  
21. (a)  
\[ f(x) = 17.5 \sin \left( \frac{\pi}{6} (x - 4) \right) + 67.5 \]  
   (b) 17.5; 12; 4 to the right; 67.5 up  
   (c) approximately 52°F  
   (d) 50°F in January; 85°F in July  
   (e) approximately 67.5°; This is the vertical translation.  
22. (a) 4 in.  (b) after \( \frac{1}{8} \) sec  
   (c) 4 cycles per sec; \( \frac{1}{4} \) sec  

9. **Angular and Linear Speed of a Point**  Suppose that point \( P \) is on a circle with radius 10 cm, and ray \( OP \) is rotating with angular speed \( \frac{\pi}{18} \) radian per sec.  
   (a) Find the angle generated by \( P \) in 6 sec.  
   (b) Find the distance traveled by \( P \) along the circle in 6 sec.  
   (c) Find the linear speed of \( P \).  

**Find each circular function value.**  
10. \( \sin \frac{3\pi}{4} \)  11. \( \cos \left( \frac{7\pi}{6} \right) \)  12. \( \tan \frac{3\pi}{2} \)  13. \( \sec \frac{2\pi}{3} \)  
14. (a) Use a calculator to approximate \( s \) in the interval \([0, \frac{\pi}{2}]\), if \( \sin s = 0.8258 \).  
   (b) Find the exact value of \( s \) in the interval \([0, \frac{\pi}{2}]\), if \( \cos s = \frac{1}{2} \).  
15. Consider the function defined by \( y = 3 - 6 \sin (2x + \frac{\pi}{2}) \).  
   (a) What is its period?  
   (b) What is the amplitude of its graph?  
   (c) What is its range?  
   (d) What is the \( y \)-intercept of its graph?  
   (e) What is its phase shift?  

**Graph each function over a two-period interval. Identify asymptotes when applicable.**  
16. \( y = -\cos 2x \)  17. \( y = -\csc 2x \)  
18. \( y = \tan \left( x - \frac{\pi}{2} \right) \)  19. \( y = -1 + 2 \sin(x + \pi) \)  
20. \( y = -2 - \cot \left( x - \frac{\pi}{2} \right) \)  

21. **(Modeling) Average Monthly Temperature**  The average monthly temperature (in °F) in Austin, Texas, can be modeled using the circular function defined by  
\[ f(x) = 17.5 \sin \left[ \frac{\pi}{6} (x - 4) \right] + 67.5, \]
where \( x \) is the month and \( x = 1 \) corresponds to January.  
(a) Graph \( f \) in the window \([1, 25]\) by \([45, 90]\).  
(b) Determine the amplitude, period, phase shift, and vertical translation of \( f \).  
(c) What is the average monthly temperature for the month of December?  
(d) Determine the maximum and minimum average monthly temperatures and the months when they occur.  
(e) What would be an approximation for the average yearly temperature in Austin?  
   How is this related to the vertical translation of the sine function in the formula for \( f \)?  

22. **(Modeling) Spring**  The height of a weight attached to a spring is  
\[ s(t) = -4 \cos 8\pi t \]
inches after \( t \) seconds.  
(a) Find the maximum height that the weight rises above the equilibrium position of \( y = 0 \).  
(b) When does the weight first reach its maximum height, if \( t \geq 0 \)?  
(c) What are the frequency and period?
Chapter 6 Quantitative Reasoning

Does the fact that average monthly temperatures are periodic affect your utility bills?

In an article entitled “I Found Sinusoids in My Gas Bill” (Mathematics Teacher, January 2000), Cathy G. Schloemer presents the following graph that accompanied her gas bill.

Your Energy Usage

<table>
<thead>
<tr>
<th>MCF</th>
<th>Average monthly temp (°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thousands of cubic feet</td>
<td></td>
</tr>
<tr>
<td></td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>75</td>
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</tr>
<tr>
<td></td>
<td>25</td>
</tr>
<tr>
<td>1998</td>
<td></td>
</tr>
</tbody>
</table>

Notice that two sinusoids are suggested here: one for the behavior of the average monthly temperature and another for gas use in MCF (thousands of cubic feet).

1. If January 1997 is represented by \( x = 1 \), the data of estimated ordered pairs (month, temperature) is given in the list shown on the two graphing calculator screens below.

   \[
   \text{L1: } \sin(\theta + 1) + L2: \sin(\theta + 2) + L3: 1
   \]

   Use the sine regression feature of a graphing calculator to find a sine function that fits these data points. Then make a scatter diagram, and graph the function.

2. If January 1997 is again represented by \( x = 1 \), the data of estimated ordered pairs (month, gas use in MCF) is given in the list shown on the two graphing calculator screens below.

   \[
   \text{L1: } 19.36 \sin(\theta) + L2: 12.69 + L3: 1
   \]

   Use the sine regression feature of a graphing calculator to find a sine function that fits these data points. Then make a scatter diagram, and graph the function.

3. Answer the question posed at the beginning of this exercise, in the form of a short paragraph.